Trigonometry

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Prerequisites And Basics

To be able to study Trigonometry sucessfully, it is recommended that students complete; Geometry, Algebra I and Algebra II prior to digging in to the course material.

It is helpful to have a graphing calculator and graph paper on hand to be able to follow along as well. If one is not available software available on sites such as http://www.graphcalc.com/ may be helpful.

Introduction

Trigonometry is an important, fundamental step in math education. From the seemingly simple shape, the right triangle, we gain tools and insight that help us in further practical as well as theoretical endeavors. The subtle mathematical relationships between the right triangle, the circle, the sine wave, and the exponential curve can only be fully understood with a firm basis in trigonometry.

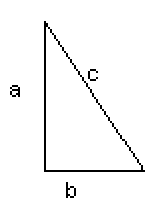
In simple terms

This page is intended as a simplified introduction to trigonometry. (This article is not always correctly formulated in mathematical language.)

Simple introduction

If you are unfamiliar with angles, where they come from, and why they are actually required, this section will help you develop your understanding.

In principle, all angles and trigonometric functions are defined on the unit circle. The term *unit* in mathematics applies to a single measure of any length. We will later apply the principles gleaned from unit measures to a larger (or smaller) scaled problems. All the functions we need can be derived from a triangle inscribed in the unit circle: it happens to be a right-angled triangle.



A Right Triangle

The center point of the unit circle will be set on a Cartesian plane, with the circle's centre at the origin of the plane — the point (0,0). Thus our circle will be divided into four sections, or quadrants.

Quadrants are always counted counter-clockwise, as is the default rotation of angular velocity ω (omega). Now we inscribe a triangle in the first quadrant (that is, where the *x*- and *y*-axes are assigned positive values) and let one leg of the angle be on the *x*-axis and the other be parallel to the *y*-axis. (Just look at the illustration for clarification). Now we let the hypotenuse (which is always 1, the radius of our *unit* circle) rotate counter-clockwise. You will notice that a new triangle is formed as we move into a new quadrant, not only because the sum of a triangle's angles cannot be beyond 180°, but also because there is no way on a two-dimensional plane to imagine otherwise.

Angle-values simplified

Imagine the angle to be nothing more than exactly the size of the triangle leg that resides on the x-axis (the cosine). So for any given triangle inscribed in the unit circle we would have an angle whose value is the distance of the triangle-leg on the x-axis. Although this would be possible in principle, it is much nicer to have a independent variable, let's call it phi, which does not change sign during the change from one quadrant into another and is easier to handle (that means it is not necessarily always a decimal number).

!!Notice that all sizes and therefore angles in the triangle are mutually directly proportional. So for instance if the x-leg of the triangle is short the y-leg gets long.

That is all nice and well, but how do we get the actual length then of the various legs of the triangle? By using translation tables, represented by a function (therefore arbitrary interpolation is possible) that can be composed by algorithms such as taylor. Those translation-table-functions (sometimes referred to as LUT, Look up tables) are well known to everyone and are known as sine, cosine and so on. (Whereas of course all the abovementioned latter ones can easily be calculated by using the sine and cosine).

In fact in history when there weren't such nifty calculators available, printed sine and cosine tables had to be used, and for those who needed interpolated data of arbitrary accuracy - taylor was the choice of word.

So how can I apply my knowledge now to a circle of any scale. Just multiply the scaling coefficient with the result of the trigonometric function (which is referring to the unit circle).

And this is also why $\cos(\varphi)^2 + \sin(\varphi)^2 = 1$, which is really nothing more than a veiled version of the pythagorean theorem: $\cos(\varphi) = a;\sin(\varphi) = b;a^2 + b^2 = c^2$, whereas the $c = 1^2 = 1$, a peculiarity of most unit constructs. Now you also see why it is so comfortable to use all those mathematical unit-circles.

Another way to interprete a angle-value would be: A angle is nothing more than a translated 'directed'-length into which the information of the actual quadrant is packed and the applied type of trigonometric function along with its sign determines the axis ('direction'). Thus something like the translation of a (x,y)-tuple into polar coordinates is a piece of cake. However due to the fact that information such as the actual quadrant is 'translated' from the sign of x and y into the angular value (a multitude of 90) calculations such as for instance the division in polar-form isn't equal to the steps taken in the non-polar form.

Oh and watch out to set the right signs in regard to the number of quadrant in which your triangle is located. (But you'll figure that out easily by yourself).

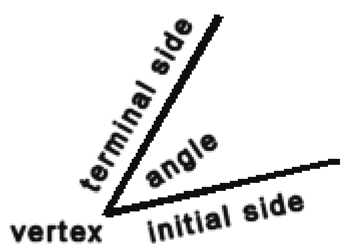
I hope the magic behind angles and trigonometric functions has disappeared entirely by now, and will let you enjoy a more in-depth study with the text underneath as your personal tutor.

Radian and Degree Measure

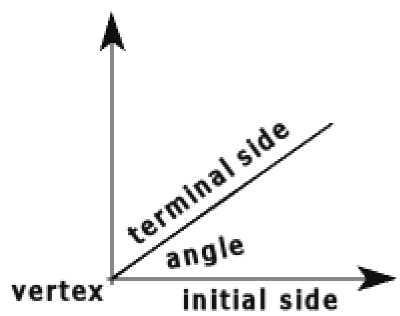
A Definition and Terminology of Angles

An angle is determined by rotating a ray about its endpoint. The starting position of the ray is called the *initial side* of the angle. The ending position of the ray is called the *terminal side*. The endpoint of the ray is called its *vertex*. *Positive angles* are generated by counter-clockwise rotation. *Negative angles* are generated by clockwise rotation. Consequently an angle has four parts: its vertex, its initial side, its terminal side, and its rotation.

An angle is said to be in *standard position* when it is drawn in a cartesian coordinate system in such a way that its vertex is at the origin and its initial side is the positive x-axis.



Definition of an Angle



An angle in standard position

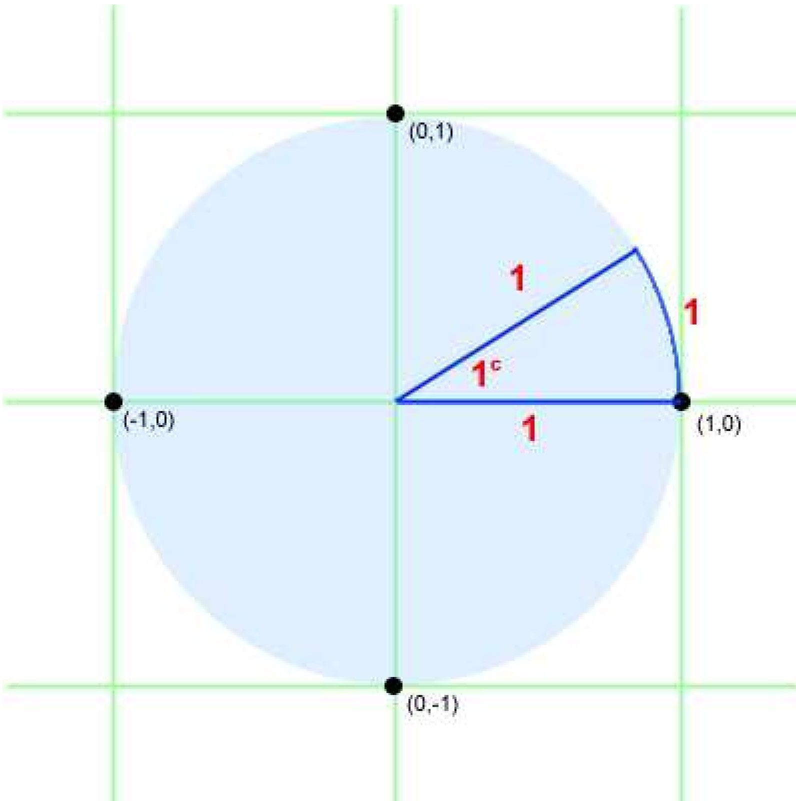
The radian measure

One way to measure angles is in radians. To signify that a given angle is in radians, a superscript c, or the abbreviation rad might be used. **If no unit is given on an angle measure, the angle is assumed to be in radians.**

$$\frac{3\pi}{2}^{c} \equiv \frac{3\pi}{2}\ \mathrm{rad.} \equiv \frac{3\pi}{2}$$

Defining a radian

A single radian is defined as the angle formed in the minor sector of a circle, where the minor arc length is the same as the radius of the circle.



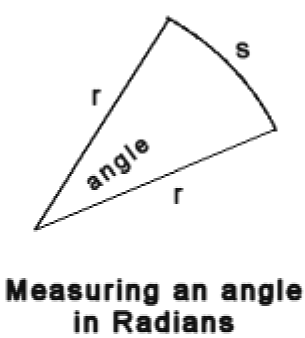
Defining a radian with respect to the unit circle.

$$1^c \approx 57.296^\circ$$

Measuring an angle in radians

The size of an angle, in radians, is the length of the circle arc *s* divided by the circle radius *r*.

$angle = \frac{s}{r}$ (rad).



Because we know the circumference of a circle to be equal to $2\pi r$, it follows that a central angle of one full counterclockwise revolution gives an arc length (or circumference) of $s = 2\pi r$. Thus 2 π radians corresponds to 360°, that is, there are 2π radians in a circle.

Converting from Radians to Degrees

Because there are 2π radians in a circle:

To convert degrees to radians:
$\theta^\circ = \theta \times \frac{\pi}{180}$ **(rad)**

To convert radians to degrees:
$\phi^c = \phi \times \frac{180^\circ}{\pi}$

Exercises

Excercise 1

Convert the following angle measurements from degrees to radians. Express your answer exactly (in terms of π).

a) 180 degrees

b) 90 degrees

c) 45 degrees

d) 137 degrees

Exercise 2

Convert the following angle measurements from radians to degrees.

5. $\frac{\pi}{3}$

6. $\frac{\pi}{6}$

7. $\frac{\pi}{12}$

8. $\frac{3\pi}{4}$

Answers

Exercise 1

a) π

b) $\frac{\pi}{2}$

c) $\frac{\pi}{4}$

d) $\frac{137\pi}{180}$

Exercise 2

a) 60°

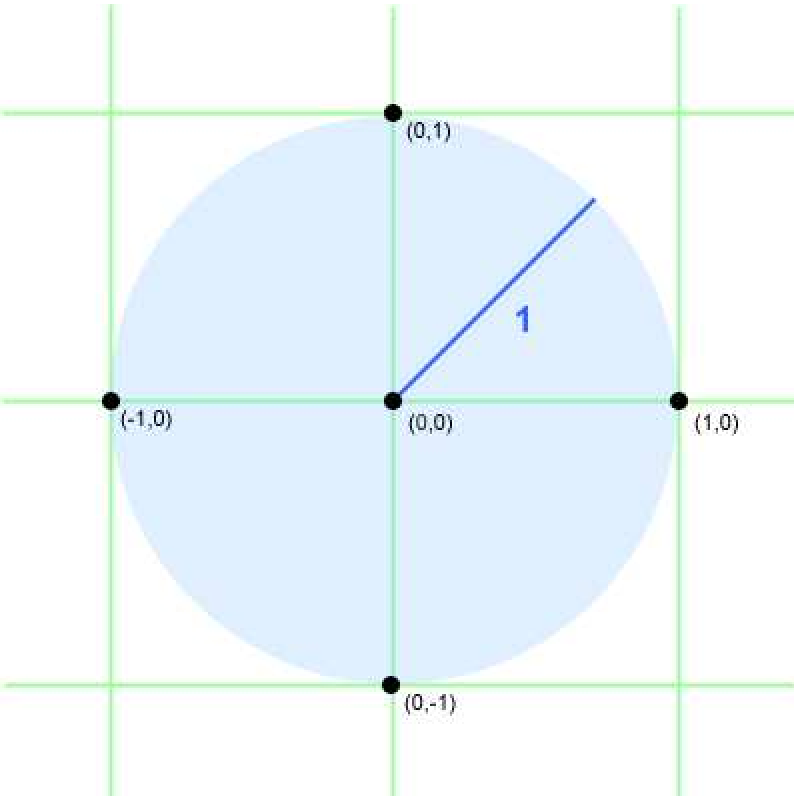
b) 30°

c) 15°

d) 135°

The Unit Circle

The Unit Circle is a circle with its center at the origin (0,0) and a radius of one unit.



The Unit Circle

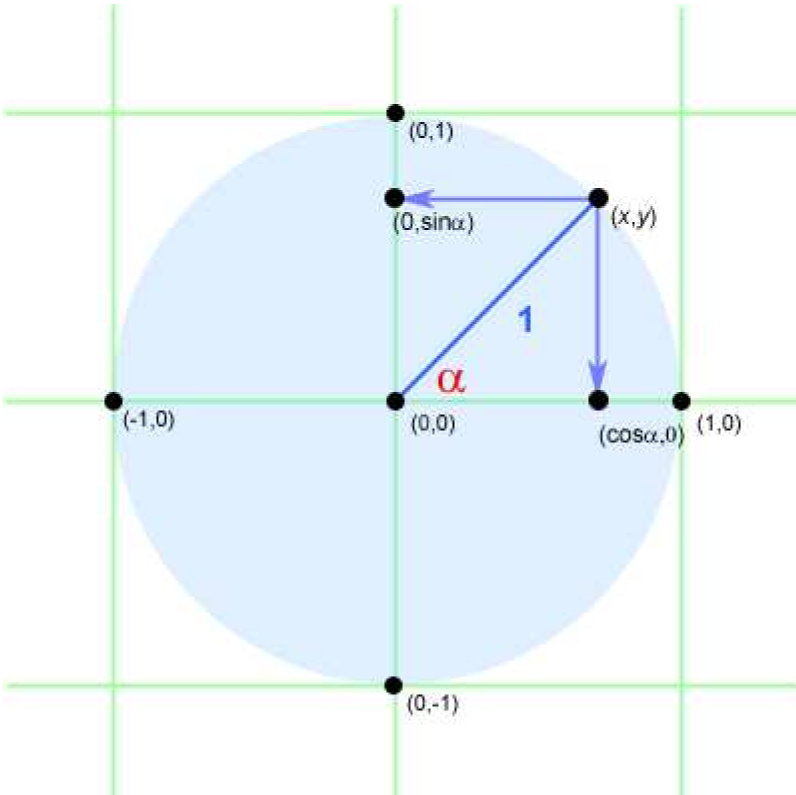
Angles are always measured from the positive x-axis (also called the "right horizon"). Angles measured counterclockwise have *positive* values; angles measured clockwise have *negative* values.

A unit circle with certain exact values marked on it is available at Wikipedia.

Trigonometric Angular Functions

Geometrically defining sin and cosine

In the unit circle shown here, a unit-length radius has been drawn from the origin to a point (x,y) on the circle.



Defining sine and cosine

A line perpendicular to the x-axis, drawn through the point (x,y), intersects the x-axis at the point with the abscissa *x*. Similarly, a perpendicular to the y-axis intersects the y-axis at the point with the ordinate *y*. The angle between the x-axis and the radius is α.

We define the basic trigonometric functions of any angle α as follows:

$$\begin{array}{rrcl} \text{Sine}: & \sin(\alpha) & = & y \\ \text{Cosine}: & \cos(\alpha) & = & x \end{array}$$

tanθ can be algebraically defined.

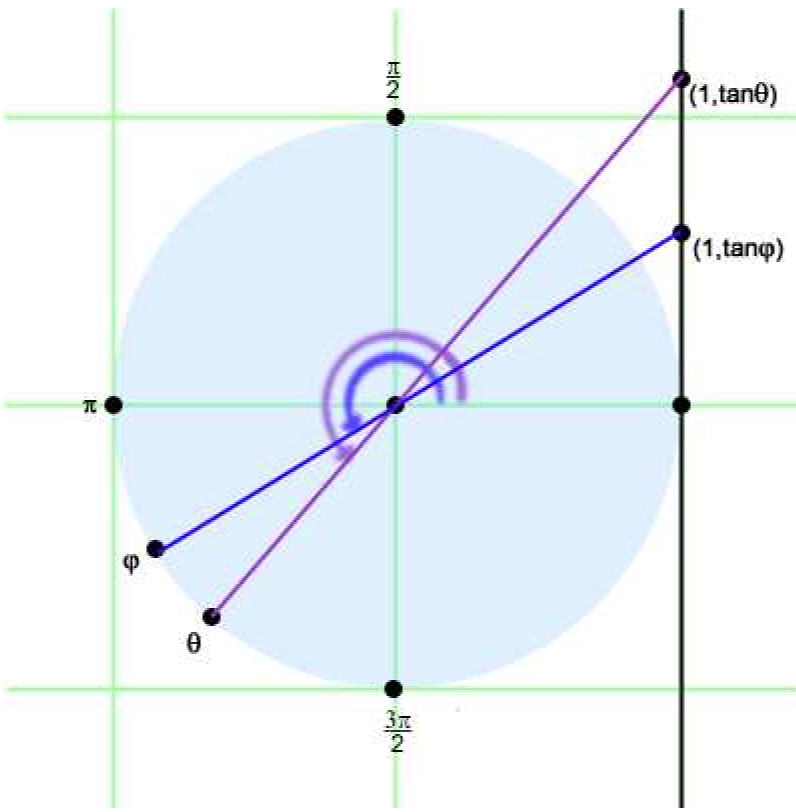
$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cos\theta \neq 0$$

$$\tan\alpha = \frac{y}{x} \qquad x \neq 0$$

These three trigonometric functions can be used whether the angle is measured in degrees or radians as long as it specified which, when calculating trigonometric functions from angles or vice versa.

Geometrically defining tangent

In the previous section, we algebraically defined tangent, and this is the definition that we will use most in the future. It can, however, be helpful to understand the tangent function from a geometric perspective.



Geometrically defining tangent

A line is drawn at a tangent to the circle: *x* = 1. Another line is drawn from the point on the radius of the circle where the given angle falls, through the origin, to a point on the drawn tangent. The ordinate of this point is called the tangent of the angle.

Domain and range of circular functions

Any size angle can be the input to sine or cosine — the result will be as if the largest multiple of 2π (or 360°) were subtracted from the angle. The output of the two functions is limited by the absolute value of the radius of the unit circle, | 1 | .

	domain	range
sine	R	$[-1, 1]$
cosine	R	$[-1, 1]$

R represents the set of all real numbers.

No such restrictions apply to the tangent, however, as can be seen in the diagram in the preceding section. The only restriction on the domain of tangent is that odd multiples of $\frac{\pi}{2}$ are undefined, as a line parallel to the tangent will never intersect it.

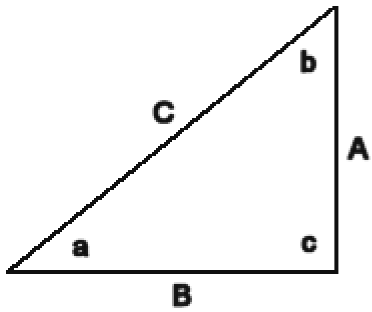
	domain	range
tangent	$R \setminus \left\{\cdots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots\right\}$	R

Applying the trigonometric functions to a right-angled triangle

If you redefine the variables as follows to correspond to the sides of a right triangle:
- x = a (adjacent)
- y = o (opposite)
- a = h (hypotenuse)

Right Angle Trigonometry

We have defined the *sine, cosine,* and *tangent* functions using the unit circle. Now we can apply them to a right triangle.



A right triangle

This triangle has *sides A* and *B*. The angle between them, *c*, is a right angle. The third side, *C*, is the *hypotenuse*. Side *A* is *opposite* angle *a*, and side *B* is *adjacent* to angle *a*.

Applying the definitions of the functions, we arrive at these useful formulas:

```
sin(a) = A / C  or opposite side over hypotenuse
cos(a) = B / C  or adjacent side over hypotenuse
tan(a) = A / B  or opposite side over adjacent side
```

Exercises: (Draw a diagram!)

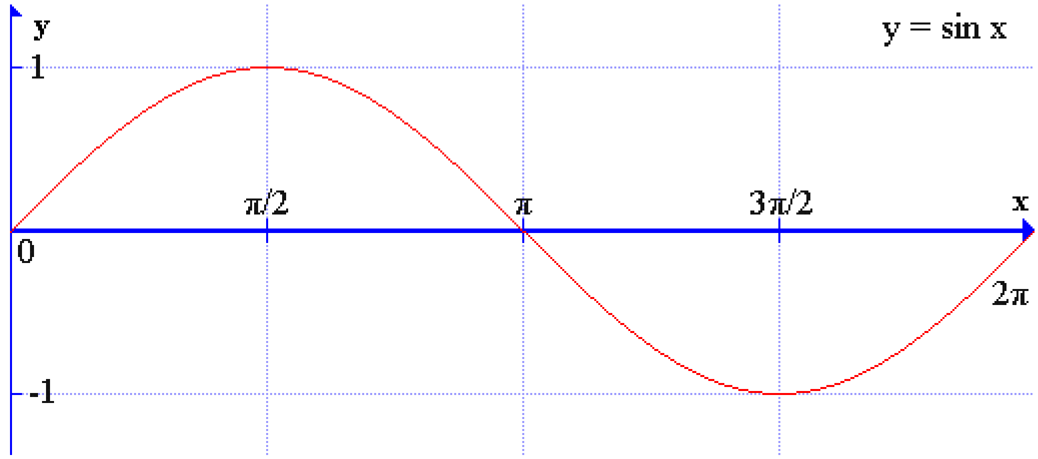
1. A right triangle has side A = 3, B = 4, and C = 5. Calculate the following:

`sin(a), cos(a), tan(a)`

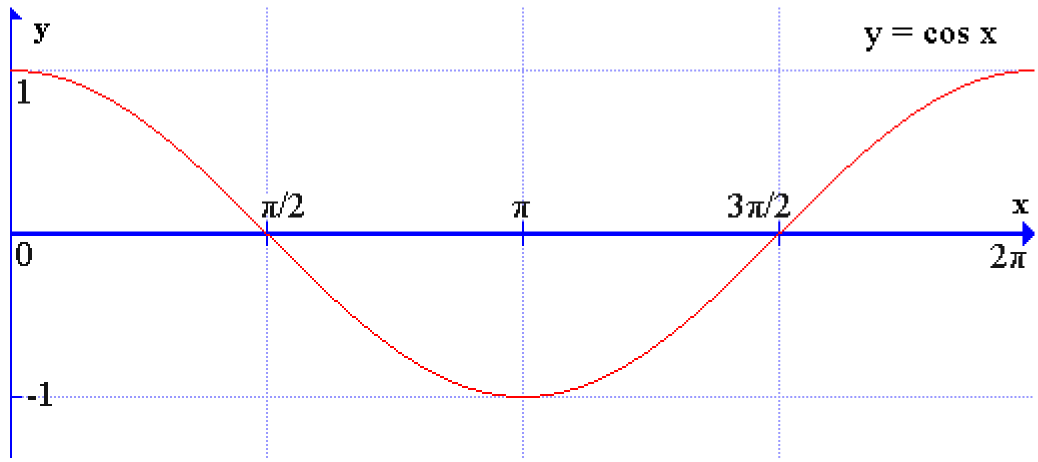
2. A different right triangle has side C = 6 and sin(a) = 0.5 . Calculate side A.

Graphs of Sine and Cosine Functions

The graph of the sine function looks like this:

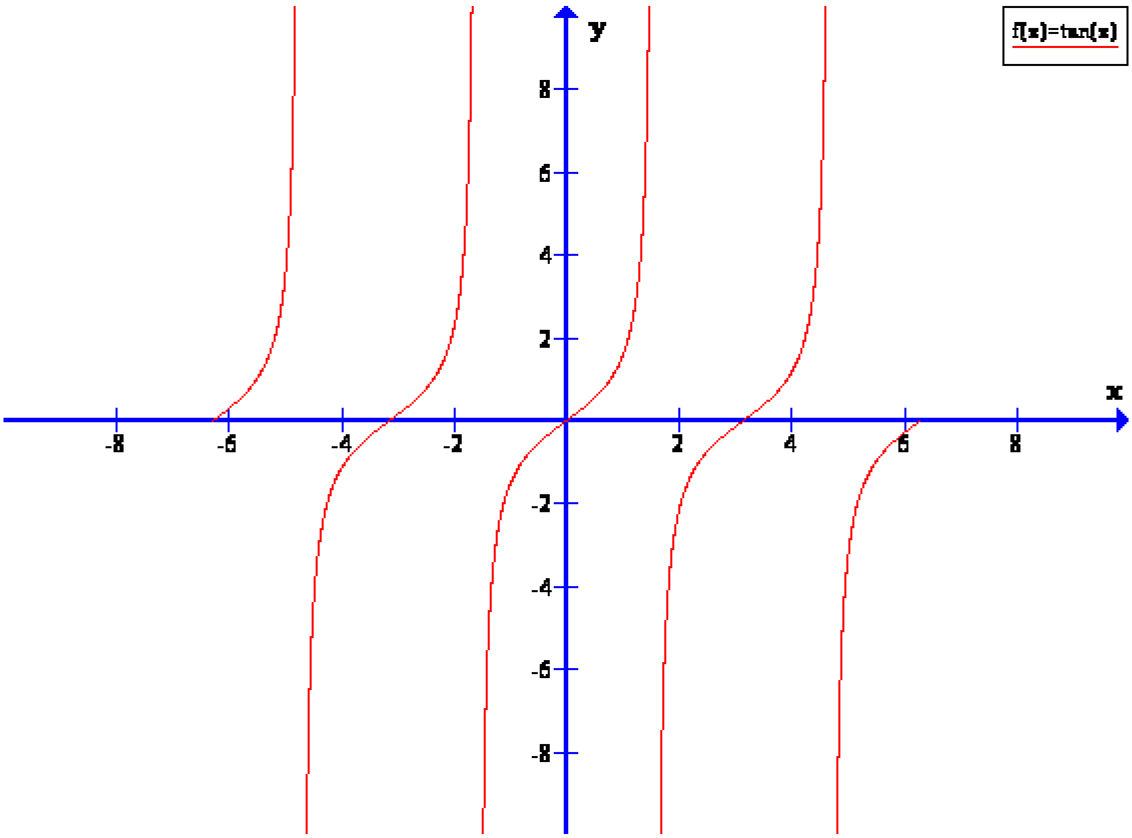


The graph of the cosine function looks like this:

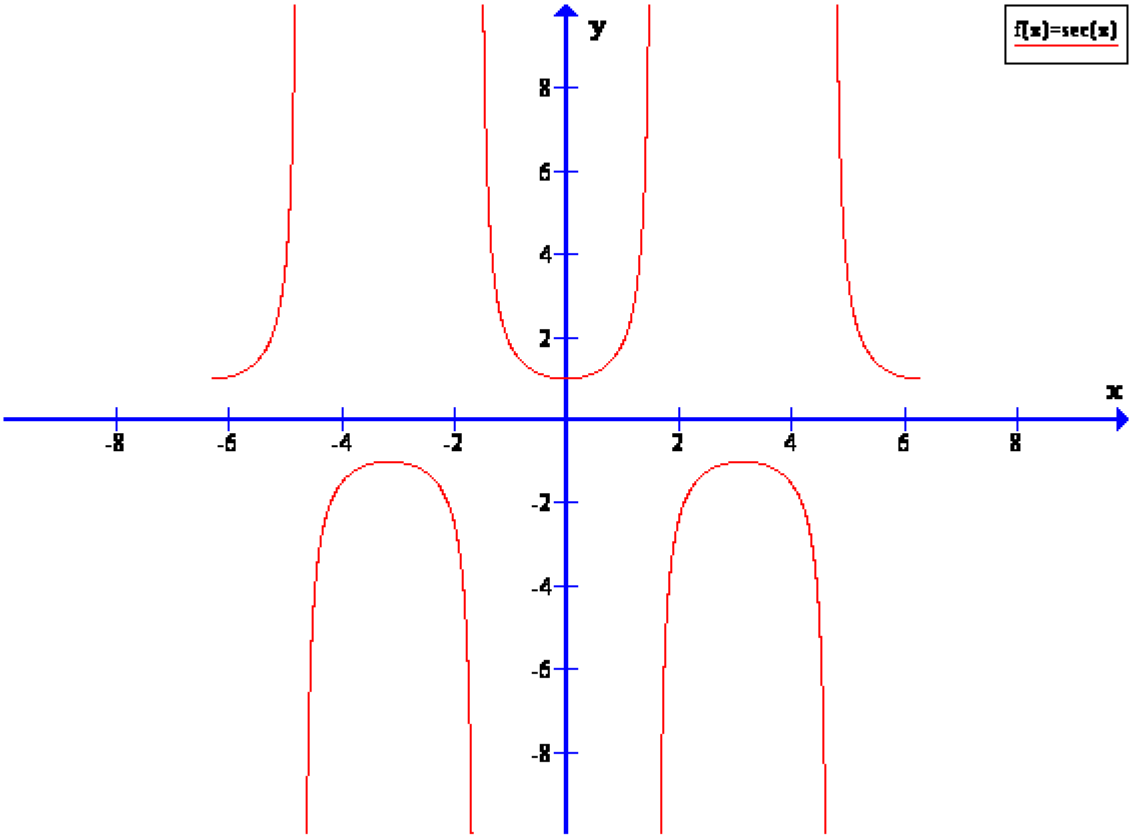


Sine and cosine are periodic functions; that is, the above is repeated for preceding and following intervals with length 2π.

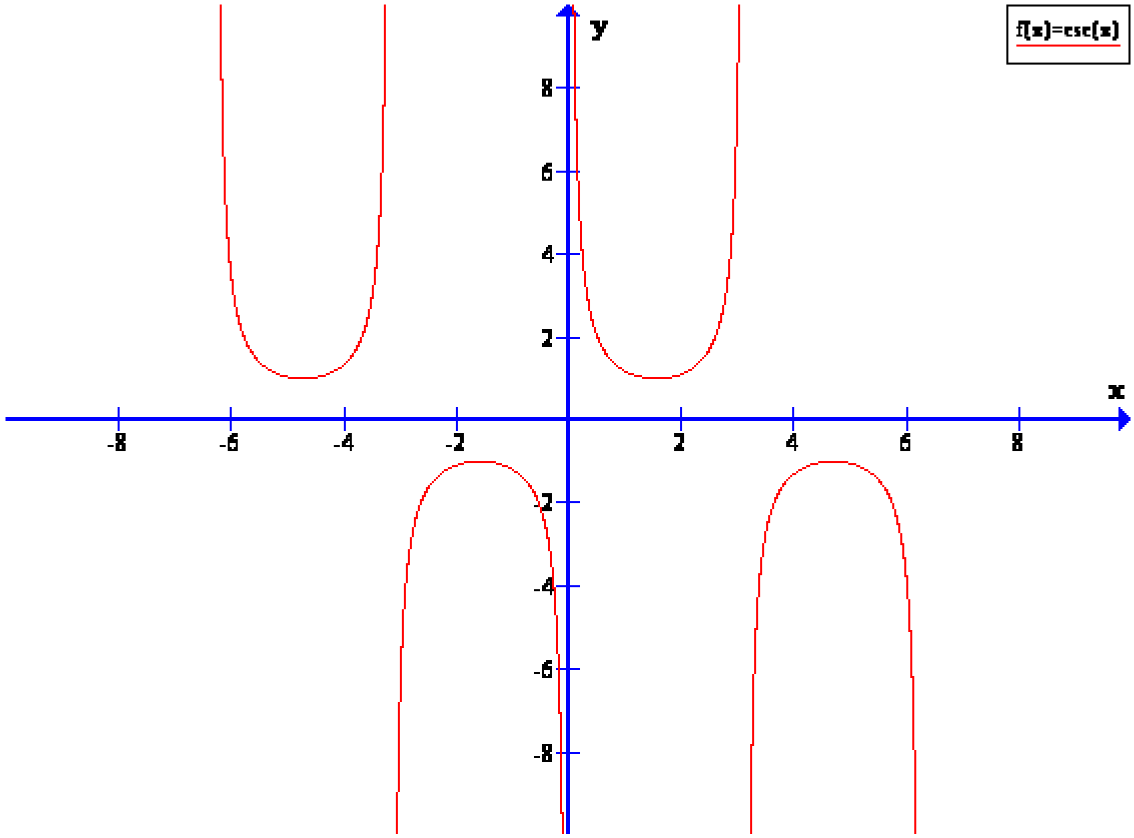
Graphs of Other Trigonometric Functions



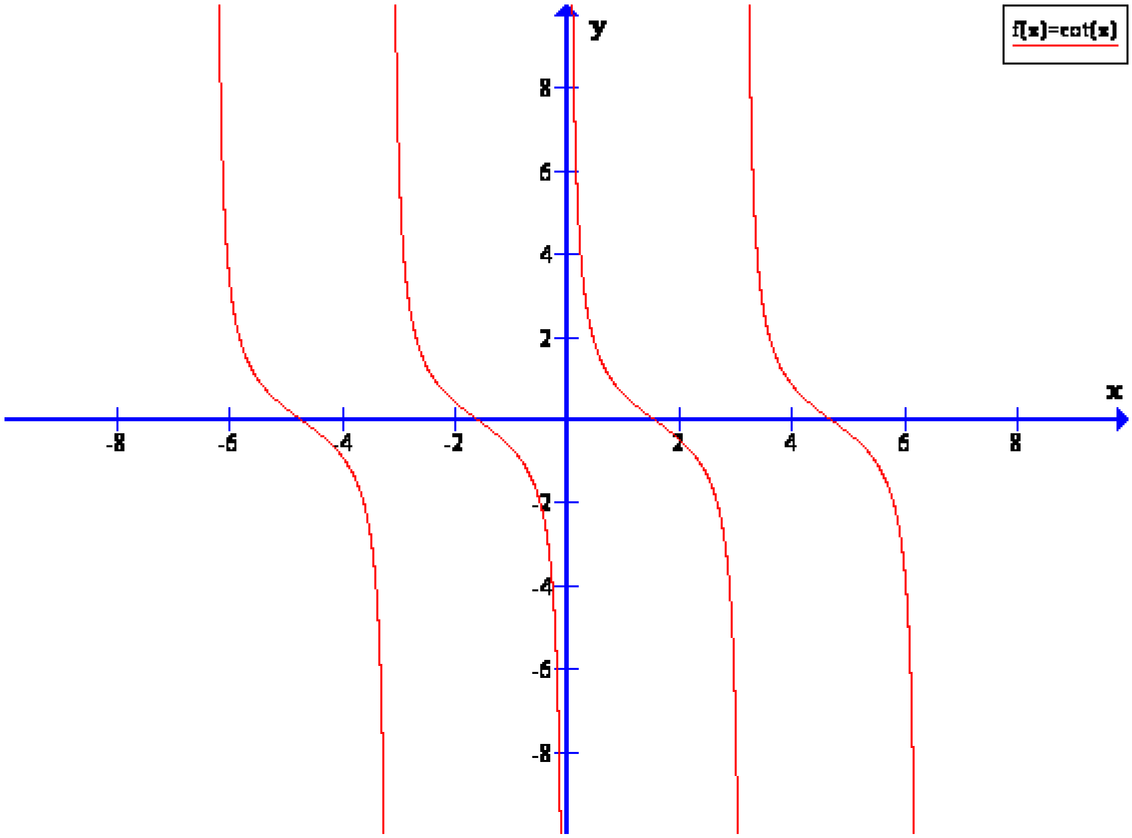
A graph of tan(*x*).



A graph of sec(*x*). sec(*x*) is defined as $\frac{1}{\cos(x)}$.



A graph of csc(x). csc(x) is defined as $\frac{1}{\sin(x)}$.



A graph of cot(x). cot(x) is defined as $\frac{1}{\tan(x)}$ or $\frac{\cos(x)}{\sin(x)}$.

Note that tan(x), sec(x), and csc(x) are unbounded.

Inverse Trigonometric Functions

The Inverse Functions, Restrictions, and Notation

While it might seem that **inverse trigonometric functions** should be relatively self defining, some caution is necessary to get an inverse *function* since the trigonometric functions are not one-to-one. To deal with this issue, some texts have adopted the convention of allowing $\sin^{-1}x$, $\cos^{-1}x$, and $\tan^{-1}x$ (all with lower-case initial letters) to indicate the inverse *relations* for the trigonometric functions and defining new functions Sin*x*, Cos*x*, and Tan*x* (all with initial capitals) to equal the original functions but with restricted domain, thus creating one-to-one functions with the inverses $\mathrm{Sin}^{-1}x$, $\mathrm{Cos}^{-1}x$, and $\mathrm{Tan}^{-1}x$. For clarity, we will use this convention. Another common notation used for the inverse functions is the "arcfunction" notation: $\mathrm{Sin}^{-1}x = \arcsin x$, $\mathrm{Cos}^{-1}x = \arccos x$, and $\mathrm{Tan}^{-1}x = \arctan x$ (the arcfunctions are sometimes also capitalized to distinguish the inverse *functions* from the inverse *relations*). The arcfunctions may be so named because of the relationship between radian measure of angles and arclength--the arcfunctions yeild arc lengths on a unit circle.

The restrictions necessary to allow the inverses to be functions are standard: $\mathrm{Sin}^{-1}x$ has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; $\mathrm{Cos}^{-1}x$ has range $[0, \pi]$; and $\mathrm{Tan}^{-1}x$ has range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (these restricted ranges for the inverses are the restricted domains of the capital-letter trigonometric functions). For each inverse function, the restricted range includes first-quadrant angles as well as an adjacent quadrant that completes the domain of the inverse function and maintains the range as a single interval.

It is important to note that because of the restricted ranges, the inverse trigonometric functions do not necessarily behave as one might expect an inverse function to behave. While $\mathrm{Sin}^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right) = \mathrm{Sin}^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ (following the expected $\mathrm{Sin}^{-1}(\sin x) = x$), $\mathrm{Sin}^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right) = \mathrm{Sin}^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$! For the inverse trigonometric functions, $f^{-1} \circ f(x) = x$ only when *x* is in the range of the inverse function. The other direction, however, is less tricky: $f \circ f^{-1}(x) = x$ for all *x* to which we can apply the inverse function.

The Inverse Relations

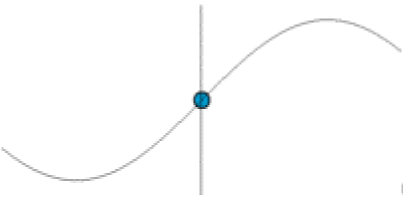
For the sake of completeness, here are definitions of the inverse trigonometric *relations* based on the inverse trigonometric *functions*:

- $\sin^{-1} x = \left\{\mathrm{Sin}^{-1}x + 2\pi n, n \in \mathbb{Z}\right\} \cup \left\{\pi - \mathrm{Sin}^{-1}x + 2\pi n, n \in \mathbb{Z}\right\}$ (the sine function has period 2π, but within any given period may have two solutions and $\sin x = \sin(\pi - x)$)

- $\cos^{-1} x = \{\pm \mathrm{Cos}^{-1} x + 2\pi n, n \in \mathbb{Z}\}$ (the cosine function has period 2π, but within any given period may have two solutions and cosine is even-- $\cos x = \cos(-x)$)
- $\tan^{-1} x = \{\mathrm{Tan}^{-1} x + \pi n, n \in \mathbb{Z}\}$ (the tangent function has period π and is one-to-one within any given period)

Applications and Models

Simple harmonic motion



Simple harmonic motion. Notice that the position of the dot matches that of the sine wave.

Simple harmonic motion (SHM) is the motion of an object which can be modeled by the following function:

$$x = A\sin\left(\omega t + \phi\right)$$

or

$$x = c_1\cos\left(\omega t\right) + c_2\sin\left(\omega t\right)$$

where $c_1 = A \sin$ Ï† and $c_2 = A \cos$ Ï†.

In the above functions, *A* is the amplitude of the motion, Ï‰ is the angular velocity, and Ï† is the phase.

The velocity of an object in SHM is

$$v = A\omega\cos\left(\omega t + \phi\right)$$

The acceleration is

$$a = -A\omega^2\sin\left(\omega t + \phi\right)$$

Springs and Hooke's Law

An application of this is the motion of a weight hanging on a spring. The motion of a spring can be modeled approximately by Hooke's Law:

$F = -kx$

where *F* is the force the spring exerts, *x* is the position of the end of the spring, and *k* is a constant characterizing the spring (the stronger the spring, the higher the constant).

Calculus-based derivation

From Newton's laws we know that $F = ma$ where m is the mass of the weight, and a is its acceleration. Substituting this into Hooke's Law, we get

$$ma = -kx$$

Dividing through by m:

$$a = -\frac{k}{m}x$$

The calculus definition of acceleration gives us

$$x'' = -\frac{k}{m}x$$

$$x'' + \frac{k}{m}x = 0$$

Thus we have a second-order differential equation. Solving it gives us

$$x = c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right) \quad \textbf{(2)}$$

with an independent variable t for time.

We can change this equation into a simpler form. By lettting c_1 and c_2 be the legs of a right triangle, with angle Ï† adjacent to c_2, we get

$$\sin\phi = \frac{c_1}{\sqrt{c_1^2 + c_2^2}}$$

$$\cos\phi = \frac{c_2}{\sqrt{c_1^2 + c_2^2}}$$

and

$$c_1 = \sqrt{c_1^2 + c_2^2}\sin\phi$$

$$c_2 = \sqrt{c_1^2 + c_2^2}\cos\phi$$

Substituting into (2), we get

$$x = \sqrt{c_1^2 + c_2^2}\sin\phi\cos\left(\sqrt{\frac{k}{m}}t\right) + \sqrt{c_1^2 + c_2^2}\cos\phi\sin\left(\sqrt{\frac{k}{m}}t\right)$$

Using a trigonometric identity, we get:

$$x = \sqrt{c_1^2 + c_2^2}\left[\sin\left(\phi + \sqrt{\frac{k}{m}}t\right) + \sin\left(\phi - \sqrt{\frac{k}{m}}t\right)\right] + \sqrt{c_1^2 + c_2^2}\left[\sin\left(\sqrt{\frac{k}{m}}t + \phi\right) + \sin\left(\sqrt{\frac{k}{m}}t - \phi\right)\right]$$

$$x = \sqrt{c_1^2 + c_2^2}\sin\left(\sqrt{\frac{k}{m}}t + \phi\right)$$ **(3)**

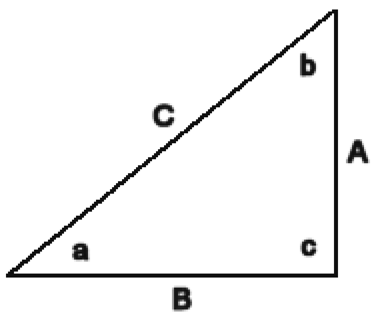
Let $A = \sqrt{c_1^2 + c_2^2}$ and $\omega^2 = \frac{k}{m}$. Substituting this into (3) gives

$$x = A\sin(\omega t + \phi)$$

Analytic Trigonometry

Using Fundamental Identities

Some of the fundamental trigometric identities are those derived from the Pythagorean Theorem. These are defined using a right triangle:



A right triangle

By the Pythagorean Theorem,

$A^2 + B^2 = C^2$ {1}

Dividing through by C^2 gives

$$\left(\frac{A}{C}\right)^2 + \left(\frac{B}{C}\right)^2 = \left(\frac{C}{C}\right)^2 = 1 \quad \{2\}$$

We have already defined the sine of a in this case as A/C and the cosine of a as B/C. Thus we can substitute these into {2} to get

$\sin^2 a + \cos^2 a = 1$

Related identities include:

$\sin^2 a = 1 - \cos^2 a$ or $\cos^2 a = 1 - \sin^2 a$

$\tan^2 a + 1 = \sec^2 a$ or $\tan^2 a = \sec^2 a - 1$

$1 + \cot^2 a = \csc^2 a$ or $\cot^2 a = \csc^2 a - 1$

Other Fundamental Identities include the **Reciprocal**, **Ratio**, and **Co-function** identities

Reciprocal identities

$$\csc a = \frac{1}{\sin a} \quad \sec a = \frac{1}{\cos a} \quad \cot a = \frac{1}{\tan a}$$

Ratio identities

$$\tan a = \frac{\sin a}{\cos a} \quad \cot a = \frac{\cos a}{\sin a}$$

Co-function identities (in radians)

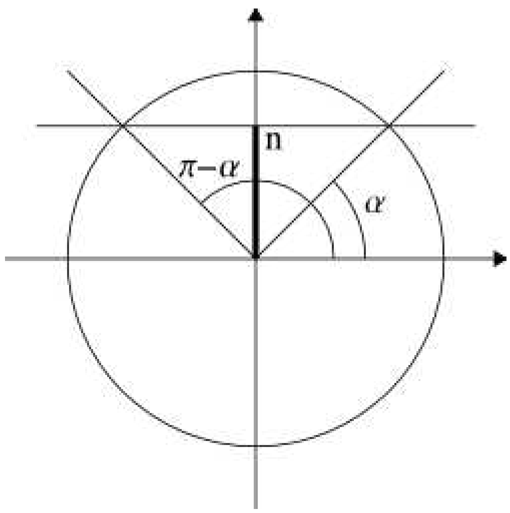
$$\cos a = \sin\left(\frac{\pi}{2} - a\right) \quad \csc a = \sec\left(\frac{\pi}{2} - a\right) \quad \cot a = \tan\left(\frac{\pi}{2} - a\right)$$

Solving Trigonometric Equations

Trigonometric equations involve finding an unknown which is an argument to a trigonometric function.

Basic trigonometric equations

sin *x* = n



n	$\sin x = n$
$\lvert n \rvert < 1$	$x = \alpha + 2k\pi$ $x = \pi - \alpha + 2k\pi$ $\alpha \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$
$n = -1$	$x = -\frac{\pi}{2} + 2k\pi$
$n = 0$	$x = k\pi$
$n = 1$	$x = \frac{\pi}{2} + 2k\pi$
$\lvert n \rvert > 1$	$x \in \varnothing$

The equation $\sin x = n$ has solutions only when n is within the interval [-1; 1]. If n is within this interval, then we need to find and α such that:

$$\alpha = \sin^{-1} n$$

The solutions are then:

$$x = \alpha + 2k\pi$$

$$x = \pi - \alpha + 2k\pi$$

Where k is an integer.

In the cases when n equals 1, 0 or -1 these solutions have simpler forms which are summarizied in the table on the right.

For example, to solve:

$$\sin \frac{x}{2} = \frac{\sqrt{3}}{2}$$

First find α:

$$\alpha = \sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3}$$

Then substitute in the formulae above:

$$\frac{x}{2} = \frac{\pi}{3} + 2k\pi$$

$$\frac{x}{2} = \pi - \frac{\pi}{3} + 2k\pi$$

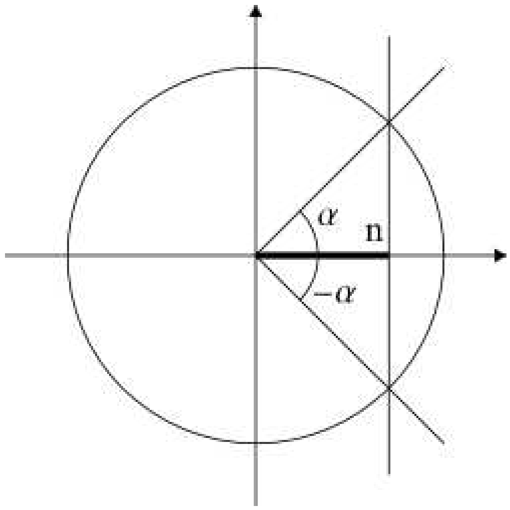
Solving these linear equations for x gives the final answer:

$$x = \frac{2\pi}{3}(1 + 6k)$$

$$x = \frac{4\pi}{3}(1 + 3k)$$

Where k is an integer.

cos *x* = n



n	$\cos x = n$
$\lvert n \rvert < 1$	$x = \pm\alpha + 2k\pi$ $\alpha \in [0; \pi]$
$n = -1$	$x = \pi + 2k\pi$
$n = 0$	$x = \frac{\pi}{2} + k\pi$
$n = 1$	$x = 2k\pi$
$\lvert n \rvert > 1$	$x \in \varnothing$

Like the sine equation, an equation of the form cos$x$ = n only has solutions when n is in the interval [-1; 1]. To solve such an equation we first find the angle α such that:

$$\alpha = \cos^{-1} n$$

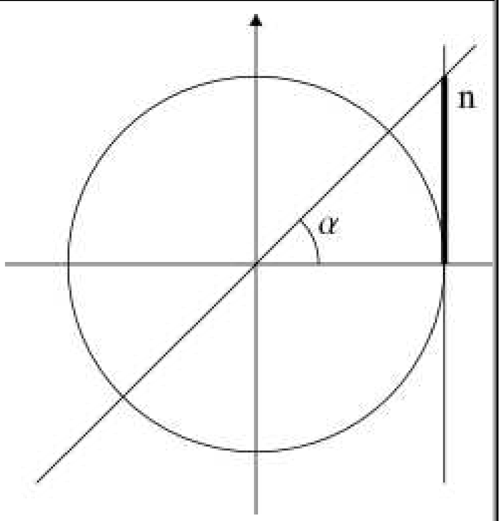
Then the solutions for x are:

$$x = \pm\alpha + 2k\pi$$

Where k is an integer.

Simpler cases with n equal to 1, 0 or -1 are summarized in the table on the right.

tan *x* = n



n	$\tan x = n$
General case	$x = \alpha + k\pi$ $\alpha \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$
$n = -1$	$x = -\frac{\pi}{4} + k\pi$
$n = 0$	$x = k\pi$
$n = 1$	$x = \frac{\pi}{4} + k\pi$

An equation of the form tan$x$ = n has solutions for any real n. To find them we must first find an angle α such that:

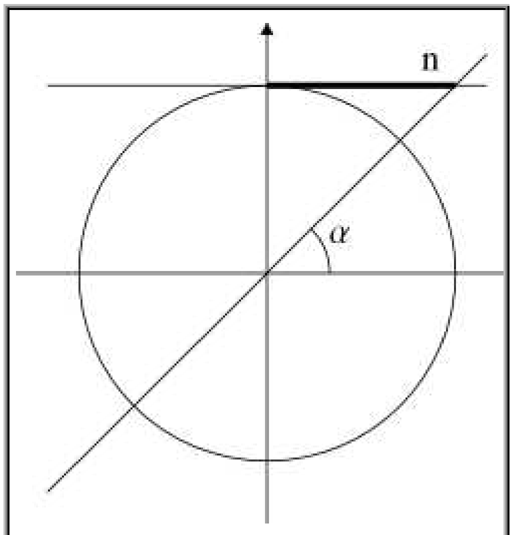
$$\alpha = \tan^{-1} n$$

After finding α, the solutions for x are:

$$x = \alpha + k\pi$$

When n equals 1, 0 or -1 the solutions have simpler forms which are shown in the table on the right.

cot *x* = n



n	$\cot x = n$
General case	$x = \alpha + k\pi$ $\alpha \in [0; \pi]$
$n = -1$	$x = -\frac{3\pi}{4} + k\pi$
$n = 0$	$x = \frac{\pi}{2} + k\pi$
$n = 1$	$x = \frac{\pi}{4} + k\pi$

The equation cot$x$ = n has solutions for any real n. To find them we must first find an angle α such that:

$$\alpha = \cot^{-1} n$$

After finding α, the solutions for x are:

$$x = \alpha + k\pi$$

When n equals 1, 0 or -1 the solutions have simpler forms which are shown in the table on the right.

csc *x* = n and sec *x* = n

The trigonometric equations csc x = n and sec x = n can be solved by transforming them to other basic equations:

$$\csc x = n \Leftrightarrow \frac{1}{\sin x} = n \Leftrightarrow \sin x = \frac{1}{n}$$

$$\sec x = n \Leftrightarrow \frac{1}{\cos x} = n \Leftrightarrow \cos x = \frac{1}{n}$$

Further examples

Generally, to solve trigonometric equations we must first transform them to a basic trigonometric equation using the trigonometric identities. This sections lists some common examples.

a* sin *x* + *b* cos *x* = *c

To solve this equation we will use the identity:

$$a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin\left(x + \alpha\right)$$

0 \\ \pi + \tan^{-1} \left(b / a\right), && \mbox{if } a

The equation becomes:

$$\sqrt{a^2 + b^2} \sin\left(x + \alpha\right) = c$$

$$\sin\left(x + \alpha\right) = \frac{c}{\sqrt{a^2 + b^2}}$$

This equation is of the form sin*x* = *n* and can be solved with the formulae given above.

For example we will solve:

$$\sin 3x - \sqrt{3} \cos 3x = -\sqrt{3}$$

In this case we have:

$$a = 1, b = -\sqrt{3}$$

$$\sqrt{a^2 + b^2} = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = 2$$

$$\alpha = \tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$$

Apply the identity:

$$2 \sin\left(3x - \frac{\pi}{3}\right) = -\sqrt{3}$$

$$\sin\left(3x - \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

So using the formulae for sin*x* = *n* the solutions to the equation are:

$$3x - \frac{\pi}{3} = -\frac{\pi}{3} + 2k\pi \Leftrightarrow x = \frac{2k\pi}{3}$$

$$3x - \frac{\pi}{3} = \pi + \frac{\pi}{3} + 2k\pi \Leftrightarrow x = \frac{\pi}{9}(6k + 5)$$

Sum and Difference Formulas

Cosine Formulas

$\cos(a + b) = \cos a \cos b - \sin a \sin b$

$\cos(a - b) = \cos a \cos b + \sin a \sin b$

Sine Formulas

sin(a + b) = sin a cos b + cos a sin b

sin(a - b) = sin a cos b - cos a sin b

sin2a = 2 sin a cos a

Tangent Formulas

tan(a +b) = (tan a + tan b)/(1 - tan a tan b)

tan(a - b) = (tan a -tan b)/(1 + tan a tan b)

Derivations

- cos(a + b) = cos a cos b - sin a sin b
- cos(a - b) = cos a cos b + sin a sin b

Using cos(a + b) and the fact that cosine is even and sine is odd, we have

```
cos(a + (-b)) = cos a cos (-b) - sin a sin (-b)
              = cos a cos b - sin a (-sin b)
              = cos a cos b + sin a sin b
```

- sin(a + b) = sin a cos b + cos a sin b

Using cofunctions we know that sin a = cos (90 - a). Use the formula for cos(a - b) and cofunctions we can write

```
sin(a + b) = cos(90 - (a + b))
           = cos((90 - a) - b)
           = cos(90 -a)cos b + sin(90 - a)sin b
           = sin a cos b + cos a sin b
```

- sin(a - b) = sin a cos b - cos a sin b

Having derived sin(a + b) we replace b with "-b" and use the fact that cosine is even and sine is odd.

```
sin(a + (-b)) = sin a cos (-b) + cos a sin (-b)
              = sin a cos b + cos a (-sin b)
              = sin a cos b - cos a sin b
```

Multiple-Angle and Product-to-sum Formulas

Multiple-Angle Formulas

- $\sin(2a) = 2 \sin a \cos a$
- $\cos(2a) = \cos^2 a - \sin^2 a = 1 - 2 \sin^2 a = 2 \cos^2 a - 1$
- $\tan(2a) = (2 \tan a)/(1 - \tan^2 a)$

Proofs for Double Angle Formulas

- $cos(a + b) = cos(a)cos(b) - sin(a)sin(b)$; a = b for cos(2a)

$cos^2(a) - sin^2(a) = cos(2a)$
$cos^2(a) - (1 - cos^2(a)) = 2cos^2 2(a) - 1 = cos(2a)$ (Note that $sin^2(x) = 1 - cos^2(x)$)

- $sin(a + b) = sin(a)cos(b) + sin(b)cos(a)$; a = b for sin(2a)

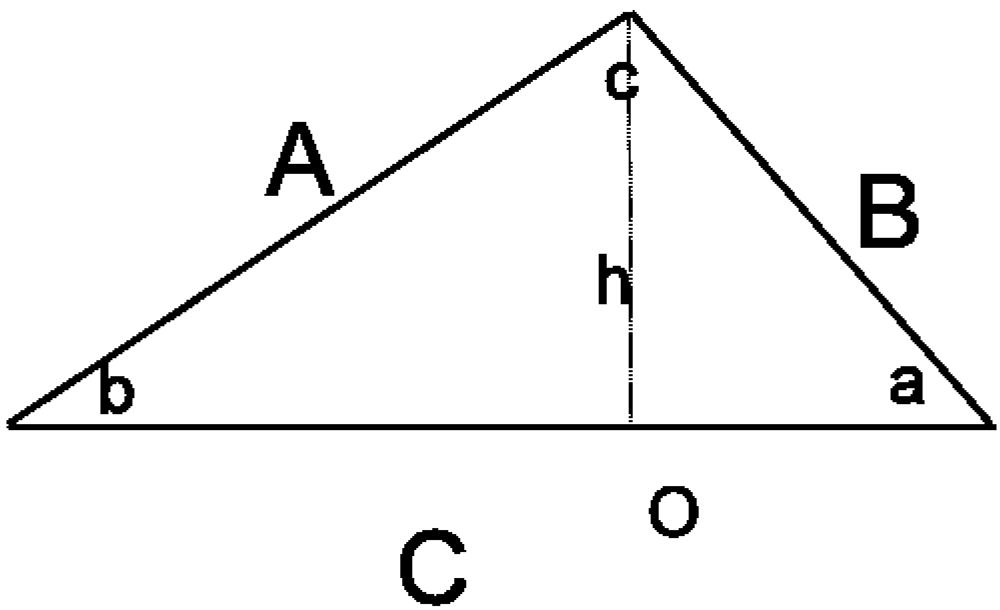
$sin(a)cos(a) + sin(a)cos(a) = 2cos(a)sin(a) = sin(2a)$

Trigonometry

Additional Topics in Trigonometry

Law of Sines

Consider this triangle:



It has three sides

- A, length *A*, opposite angle *a* at vertex a
- B, length *B*, opposite angle *b* at vertex b
- C, length *C*, opposite angle *c* at vertex c

The perpendicular, oc, from line ab to vertex c has length *h*

The Law of Sines states that:

$$\frac{A}{\sin a} = \frac{B}{\sin b} = \frac{C}{\sin c}$$

The law can also be written as the reciprocal:

$$\frac{\sin a}{A} = \frac{\sin b}{B} = \frac{\sin c}{C}$$

Proof

The perpendicular, oc, splits this triangle into two right-angled triangles. This lets us calculate *h* in two different ways

- Using the triangle cao gives

$$h = B \sin a$$

- Using the triangle cbo gives

$$h = A \sin b$$

- Eliminate h from these two equations

$$A \sin b = B \sin a$$

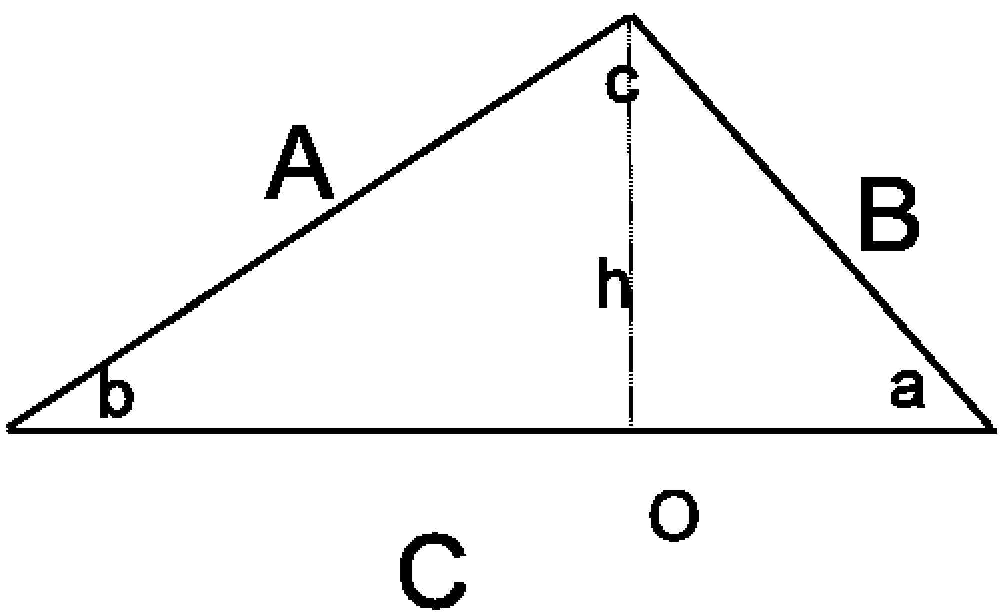
- Rearrange

$$\frac{A}{\sin a} = \frac{B}{\sin b}$$

By using the other two perpendiculars the full law of sines can be proved.

Law of Cosines

Consider this triangle:



It has three sides

- a, length *a*, opposite angle *A* at vertex A
- b, length *b*, opposite angle *B* at vertex B
- c, length *c*, opposite angle *C* at vertex C

The perpendicular, oc, from line ab to vertex c has length *h*

The Law Of Cosines states that:

$$a = \sqrt{b^2 + c^2 - 2bc \cdot \cos A}$$

$$b = \sqrt{a^2 + c^2 - 2ac \cdot \cos B}$$

$$c = \sqrt{a^2 + c^2 - 2ab \cdot \cos C}$$

Proof

The perpendicular, oc, divides this triangle into two right angled triangles, aco and bco.

First we will find the length of the other two sides of triangle aco in terms of known quantities, using triangle bco.

$h = a \sin B$

Side c is split into two segments, total length *c*.

ob, length $c \cos B$

ao, length $c - a \cos B$

Now we can use Pythagoras to find *b*, since $b^2 = \text{"ao"}^2 + h^2$

$$\begin{aligned} b^2 &= (c - a\cos B)^2 + a^2 \sin^2 B \\ &= c^2 - 2ac\cos B + a^2\cos^2 b + a^2 \sin^2 B \\ &= a^2 + c^2 - 2ac\cos B \end{aligned}$$

The corresponding expressions for *a* and *c* can be proved similarly.

Vectors and Dot Products

Consider the vectors **U** and **V** (with respective magnitudes |**U**| and |**V**|). If those vectors enclose an angle θ then the dot product of those vectors can be written as:

$$\mathbf{U} \cdot \mathbf{V} = |\mathbf{U}||\mathbf{V}| \cos(\theta)$$

If the vectors can be written as:

$$\mathbf{U} = (U_x, U_y, U_z)$$

$$\mathbf{V} = (V_x, V_y, V_z)$$

then the *dot product* is given by:

$$\mathbf{U} \cdot \mathbf{V} = U_x V_x + U_y V_y + U_z V_z$$

Trigonometric Form of the Complex Number

$$z = a + bi = r\,(\cos\phi \; + i\sin\phi)$$

where

- *i* is the Imaginary Number $\left(\sqrt{i} \; = -1\right)$
- the modulus $r = \operatorname{mod}(z) = |z| = \sqrt{a^2 + b^2}$
- the argument $\phi = \arg(z)$ is the angle formed by the complex number on a polar graph with one real axis and one imaginary axis. This can be found using the right angle trigonometry for the trigonometric functions.

This is sometimes abbreviated as $r\,(\cos\phi \; + i\sin\phi) = r\mathrm{cis}\phi$ and it is also the case that $r\mathrm{cis}\phi = re^{i\phi}$ (provided that φ is in radians).

Trigonometry References

Trigonometric Formula Reference

The principal identity in trigonometry is:

$\sin^2\theta + cos^2\theta = 1$

All trigonometric function are 2π periodic:

$\sin\theta = \sin(\theta + 2\pi)$

$\cos\theta = \cos(\theta + 2\pi)$

$\tan\theta = \tan(\theta + 2\pi)$

$\cot\theta = \cos(\theta + 2\pi)$

$\csc\theta = \cos(\theta + 2\pi)$

$\sec\theta = \cos(\theta + 2\pi)$

Formulas involving sums of angles are as follows:

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

Substituting $\alpha = \beta$ gives the **double angle formulae**

Trigonometric Identities Reference

Pythagoras

1. $\sin^2(x) + \cos^2(x) = 1$
2. $1 + \tan^2(x) = \sec^2(x)$
3. $1 + \cot^2(x) = \csc^2(x)$

These are all direct consequences of Pythagoras's theorem.

Sum/Difference of angles

1. $\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$

2. $\sin(x \pm y) = \sin(x)\cos(y) \pm \sin(y)\cos(y)$

3. $\tan(x \pm y) = \dfrac{\tan(x) \pm \tan(y)}{1 \mp \tan(x)\tan(y)}$

Product to Sum

1. 2sin(*x*)sin(*y*) = cos(*x* - *y*) - cos(*x* + *y*)
2. 2cos(*x*)cos(*y*) = cos(*x* - *y*) + cos(*x* + *y*)
3. 2sin(*x*)cos(*y*) = sin(*x* - *y*) + sin(*x* + *y*)

Sum and difference to product

1. *A*sin(*x*) + *B*cos(*x*) = *C*sin(*x* + *y*)

where $C = \sqrt{A^2 + B^2}$ and $y = \tan^{-1}(B/A)$

1. $\sin\alpha + \sin\beta = 2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$
2. $\sin\alpha - \sin\beta = 2\cos\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}$
3. $\cos\alpha + \cos\beta = 2\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$
4. $\cos\alpha - \cos\beta = -2\sin\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}$

Double angle

1. $\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$
2. $\sin(2x) = 2\sin(x)\cos(x)$
3. $\tan(2x) = \dfrac{2\tan(x)}{1 - \tan^2(x)}$

These are all direct consequences of the sum/difference formulae

Half angle

1. $\cos(\frac{x}{2}) = \pm\sqrt{\frac{1+\cos(x)}{2}}$
2. $\sin(\frac{x}{2}) = \pm\sqrt{\frac{1-\cos(x)}{2}}$
3. $\tan(\frac{x}{2}) = \frac{1-\cos(x)}{\sin(x)} = \frac{\sin(x)}{1+\cos(x)} = \pm\sqrt{\frac{1-\cos(x)}{1+\cos(x)}}$

In cases with $\pm$, the sign of the result must be determined from the value of $\frac{x}{2}$. These derive from the cos(2*x*) formulae.

Power Reduction

1. $\sin^2\theta = \frac{1-\cos 2\theta}{2}$
2. $\cos^2\theta = \frac{1+\cos 2\theta}{2}$
3. $\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$

Even/Odd

1. $\sin(-\theta) = -\sin(\theta)$
2. $\cos(-\theta) = \cos(\theta)$
3. $\tan(-\theta) = -\tan(\theta)$
4. $\csc(-\theta) = -\csc(\theta)$
5. $\sec(-\theta) = \sec(\theta)$
6. $\cot(-\theta) = -\cot(\theta)$

Calculus

1. $\frac{d}{dx}[\sin x] = \cos x$
2. $\frac{d}{dx}[\cos x] = -\sin x$
3. $\frac{d}{dx}[\tan x] = \sec^2 x$
4. $\frac{d}{dx}[\sec x] = \sec x \tan x$
5. $\frac{d}{dx}[\csc x] = -\csc x \cot x$
6. $\frac{d}{dx}[\cot x] = -\csc^2 x$

Natural Trigonometric Functions of Primary Angles

θ (radians)	sinθ	cosθ	tanθ	cotθ	secθ	cscθ	θ (degrees)
0	0	1	0	undef	1	undef	0°
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	30°
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$	45°
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	60°
$\frac{\pi}{2}$	1	0	undef	0	undef	1	90°
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	- 2	$\frac{2}{\sqrt{3}}$	120°
$\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	- 1	- 1	$-\sqrt{2}$	$\sqrt{2}$	135°
$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	2	150°
π	0	- 1	0	undef	- 1	undef	180°
$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	- 2	210°
$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	- 1	- 1	$-\sqrt{2}$	$-\sqrt{2}$	225°
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	- 2	$-\frac{2}{\sqrt{3}}$	240°
$\frac{3\pi}{2}$	- 1	0	undef	0	undef	- 1	270°
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	2	$-\frac{2}{\sqrt{3}}$	300°
$\frac{7\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	- 1	- 1	$\sqrt{2}$	$-\sqrt{2}$	315°
$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$\frac{2}{\sqrt{3}}$	- 2	330°
2π	0	1	0	undef	1	undef	360°

Note: some values in the table are given in forms that include a radical in the denominator--this is done both to simplify recognition of reciprocal pairs and because the form given in the table is *simpler* in some sense.

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