Chapter 2
Percent and Conversion

Our goal in the first three chapters of this text is to help students gain an intuitive sense of comparison of quantities using models and partial tables. As students become more fluent with the ideas, they will develop more efficient algorithmic approaches. With a solid understanding of number, they will be better able to determine the appropriateness of their answer. The progression through these chapters is designed to reach this goal effectively, rather than to pin down algorithms and procedures.

In this chapter we use the concept of ratio to understand, employ and compare the various representations of numbers (points on the line, decimals, fractions, percents), and the arithmetic operations on numbers. This works both ways: the rules of arithmetic help us to understand how ratios compare and combine.

We begin the first section by focusing on the role units play in applying the idea of ratio. 45:60 is meaningless without context. This particular ratio is equivalent to 3:4 which is as meaningless without context. However, if we attach the context of miles:minutes then 45 miles:60 minutes not only has meaning to us, we can manipulate the ratio in various ways to answer questions. For example, we might convert it to 45 miles per hour, or 3/4 mile per minute. Converting all or part of a ratio's units will be addressed later in the chapter. Some conversions arose automatically in Chapter 1; here we explore the value of conversions more deeply. For now, it suffices to say that these are all equivalent statements, each one stressing a different choice of units.

Percent is a common way to represent ratios. The word, percent, comes from the Latin: *per* means "for each" and *cent* means "hundred," so percent is literally *for each hundred.* The symbol for *percent* is %. So, 70% says 70 for each hundred, and can be expressed by the ratio 70:100, or the fraction 70/100. Many students benefit from models as they learn about percents.

One common model for percentage is an array of 100 squares in a 10 by 10 grid, as in the image below. Since 1% is 1/100 (of the whole), when we look at the large square as the whole, each small square corresponds to 1%. Note that shaded portions do not have to touch to be included in the percent. Figure 1 below shows 1%, 70%, and 68%, respectively.

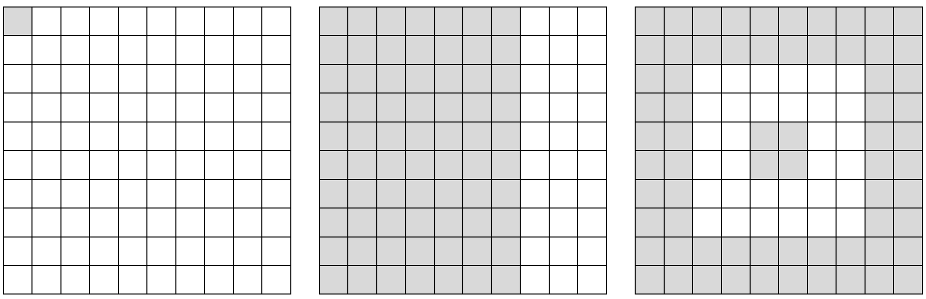


Figure 1

Another common model is the bar (tape) model. This is frequently used when the percent in question is a multiple of a divisor of 100 (such as multiples of 5%, 10%, 25%, etc.). Since 70% is a multiple of 10%, we could use a bar (or tape) cut into 10 equal-sized pieces.

$\longleftarrow$ 100% = 10 × 10% $\longrightarrow$

10%	10%	10%	10%	10%	10%	10%	10%	10%	10%

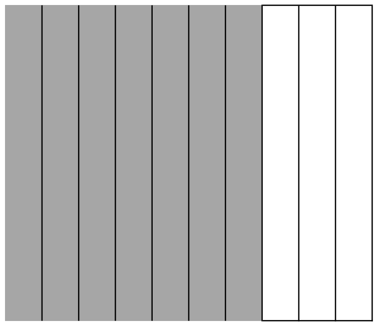
In this way, each piece represents 1/10, or equivalently 10/100 = 10%, of the whole. Therefore, to represent 70%, we shade in 7 of the 10 pieces, as demonstrated below.

10%	10%	10%	10%	10%	10%	10%	10%	10%	10%

Figure 2

Note that if we were to take the 10 by 10 grid of the central image in Figure 1 and remove the horizontal lines inside the square (as in the figure to the right), then we would have Figure 2 somewhat elongated. This observation helps students make the transition between different representations of the same fraction and their corresponding percent. This is simply the statement that

$$\frac{7}{10} = \frac{70}{100} = 70\%.$$



Conversion of fractions to percentages provides a uniform way to standardize ratios and fractions so that we can compare them. For example, the comparison of 8/25 with 38/125 may not be self-evident, but the comparison of the equivalent percents (32% and 30.4%, respectively) is much easier (as is the comparison of equivalent fractions: 40/125 vs. 38/125). Percentages are useful in giving answers to questions like the following:

a. Which is larger: 5/8 or 3/5?

b. How much larger than 5/6 is 6/7?

c. On exam one I got 45 of 50 questions correct, and on the second exam, I got 68 of 75 questions right. Did I improve my performance?

d. In the first six months, the Ravson baby grew from 9 lbs to 14.5 lbs, and the Ikemba baby grew from 6.3 lbs to 10 lbs. Which baby has shown the greater growth?

e. At store A, they knocked $40 off the price of a $145 pair of shoes, and at store B, they have reduced their price of $160 for the same pair of shoes by 35%. Which is now the better price?

See Example 10 for a discussion of these questions.

In section 2 we focus on the division of numbers by fractions. This leads to the rule that, for positive numbers A, a, b:

$$A \div \frac{a}{b} = A \times \frac{b}{a} .$$

It is fine for students to use this rule, but it is more important that they understand why it works. This will help them in practical situations.

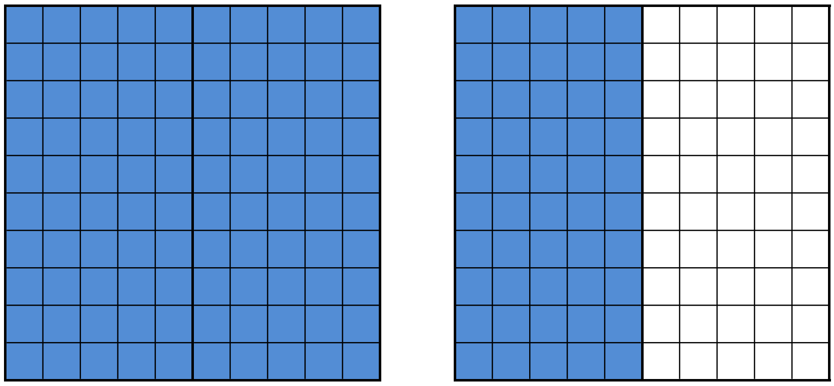
In section 3 we discuss conversions of units in expressing ratios. For example, a jelly recipe calling for 1.5 lbs of sugar to 8 cups of berries, can be expressed as 3/4 lb of sugar per quart (4 cups) of berries. A good runner runs 100 yards in 12 seconds; this is a pace of 500 yards per minute, or a mile in a little over three minutes. The first conversion is useful if one is producing large quantities of berries (so for 60 quarts of berries we need 45 lbs of sugar), and the second is useful if one wants to compare the speed of sprinters with that of long distance runners.

Section 1. Ratio out of 100 (Percents)

Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent. 6.RP.3c

The importance of units

Example 1. In the figure below, what ratio is being modeled by the shaded portion in this graphic?



Solution. Answers will vary: 3:4, 3:1, 150:200, 1:0.5, and more. For each answer we can envision a context for which that answer is correct. To do that, we provide this context for the problem: suppose I am washing the windows in my study. There are two windows in my study, each has two panes, and each pane is paneled by small squares: 5 squares across, and 10 down. The part that has already been washed is in blue.

- 3:4. Of four panes in my study, I have washed 3 out of 4. This is the ratio *washed : total*.
- 3:1. Of four panes in my study, I have washed 3. This is the ratio *washed : unwashed*.
- 150:200. Here we are counting by panels: I have washed 150 panels out of 200. The ratio 150:50 could also come up, showing the ratio *work done : work to do* in terms of panels.
- 1:0.5. This is the ratio of *amount of first window washed : amount of second window washed.*

The distinction between the first and third responses and the second and fourth is crucial, and has to be articulated whenever a ratio is discussed. Responses 1 and 3 are *part to whole* and responses 2 and 4 are *part to part*. In most contexts, a ratio is given as part to whole (George ate 6/8 of the pizza), but sometimes as part to part (George ate six pieces of the pizza, and Carl ate the other two pieces). In racing contexts a part:whole ratio describes the chances of winning, and a part:part ratio (called *odds*), compares those chances between two players.

Example 2. In the figure of Example 1, what number is being modeled by the shaded portion in this graphic?

Solution. Answers will vary: 3, 3/2, 3/4, 150, 150/200, 150/100. For each answer we can envision a context for which that answer is correct. The context will answer the question: 3 of what?; 3/2 of what?;150 of what? That is, we have to select what, in this diagram, models *one unit*.

- 3: The unit is *pane*. I have washed 3 panes. The shaded portion represents "3."
- 3/2: The unit is *window*. I have washed one and a half (3/2) windows.
- 3/4: The unit is *job*. I have washed 3 panes of 4. My job is 3/4 done.
- 150: The unit is *panel*. I have washed 150 panels.
- 150/200: The unit is *panel*, and the illustration shows the ratio of the amount I've washed to the full job - in panels.
- 150/100: Here two units have been selected: that of panels and that of windows. The number models the number of windows done, in terms of panels.

These responses have been justified in terms of the unit selected. What has not been articulated is the goal served by the graphic, and it is that goal that determines the measures and the units. Am I interested in finishing the job, or assessing my accomplishments? Does that assessment sound better in terms of three panes washed, or 150 panels washed? Am I interested in how it sounds, or in the end result?

Example 3. Boris ran the 100 meters in 12 seconds, and Francesca ran the 5 km race in 12 minutes. Who is faster?

Solution. First of all, sprinters should not be compared to distance runners. We should expect Boris to be the faster runner - but don't ask him to run a 5 km race. The issue here is to compare the speeds: the rates at which Boris and Francesca run their races *expressed in the same units*. Let's fix the unit of time in minutes. Boris runs 100 meters in 12 seconds - that is one-fifth of a minute. So, at that same rate, he can run 500 meters in a minute. Francesca runs 5000 meters in 12 minutes (note the change of unit from km to meter). We can say that (at the pace he runs the sprint) Boris runs 5000 meters in 10 minutes. Thus Boris is faster. Or, we can use *minute* as the unit of comparison: Boris runs 500 meters in a minute, and Francesca 5000 m./12 min. = 416.7 meters per minute. Boris is still faster!

Even if ratios are given in the abstract, in order to compare them, it is necessary to find a commonality among them (that is, a common unit). Consider this:

Example 4. Of three containers A, B, C, we are told that the ratios of the capacities of these containers are $A : B :: 5 : 4$ and $A : C :: 6 : 5$. Order the containers as to size.

Here we are given two ratios and asked to make a deduction about a third ratio. What is given can be expressed by fractions as follows:

(1) $$\frac{A}{B} = \frac{5}{4} \qquad \frac{A}{C} = \frac{6}{5}\ .$$

Clearly, the content of A is greater than the content of each of the containers B and C; but the question is: of B and C, which has the greater capacity?

Solution. #1. To compare fractions, we replace the fractions on the right of (1) by equivalent fractions so that the comparison is obvious. Since, in (1) the numerators on the left are both A, we replace the fractions on the right so that they have the same numerators:

(2) $$\frac{A}{B} = \frac{30}{24} \qquad \frac{A}{C} = \frac{30}{25}\ .$$

We conclude, since $25 > 24$, that the capacity of C is closer to the capacity of A than is that of B and thus that C has more capacity than B.

Solution. #2. The key to this problem is the selection of a unit. For example: "Suppose that A contains 10 gallons, then what is the capacity of B and that of C?" We turn equations (1) into equations relating gallons by replacing A by 10, giving us

$$\frac{10}{B} = \frac{5}{4} \qquad \frac{10}{C} = \frac{6}{5} .$$

We can solve the first equation for B, getting $B = 40/5 = 8$ gallons. Now solve for C in the second equation: $C = 50/6 = 8\frac{1}{3}$ gallons. So, C has the greater capacity than B.

Solution. #3. Students who are familiar with algebra may see how to use equations (1) to find the ratio of B to C, as follows. Change the first problem to read

$$\frac{B}{A} = \frac{4}{5} \qquad \frac{A}{C} = \frac{6}{5} .$$

Now multiply both sides of the equations to get:

$$\frac{B}{C} = \frac{B}{A}\frac{A}{C} = \frac{4}{5}\frac{6}{5} = \frac{24}{25} ,$$

from which we conclude that C has greater capacity than B.

Solution 3 hopefully won't arise, as it uses algebra in a way in which students are not yet well versed.. Solutions 1 and 2 are based on the realization that, to compare ratios, we need to impose a commonality on the ratios. In 2 we chose "gallons," and arbitrarily assigned to A a capacity of 10 gallons. It should be noticed that "10" is arbitrary and could have been any number: the comparison will always produce the same result. In Solution 1 we replace the ratios 5:4 and 6:5 with the equivalent ratios 30:24 and 30:25, so that the direct comparison of denominators tells us what we want to know.

Percentage

The usefulness of percentages is to pick a uniform common unit, independent of the numbers particular to the problem, that is, to take the unit to be 100. To apply this to Example 2, we first rewrite the ratios as $B : A :: 4 : 5$, $C : A :: 5 : 6$. Now, replacing each ratio by an equivalent ratio which, in each case the second term is 100, we get, instead of the equations (2), the following:

(3) $$\frac{B}{A} = \frac{80}{100} \qquad \frac{C}{A} = \frac{83.3}{100} .$$

This is to be read as saying that B is 80 percent of A and C is 83.3 percent of A, so C has greater capacity than A.

We define percentage as follows:

> Given positive numbers A and B, we say that A is P percent of B if $A : B :: P : 100$, that is, A is to B as P is to 100. In terms of fractions:
>
> $$\frac{A}{B} = \frac{P}{100} .$$

Another way of writing this relationship is to algebraically express the statement "A is P percent of B" as:

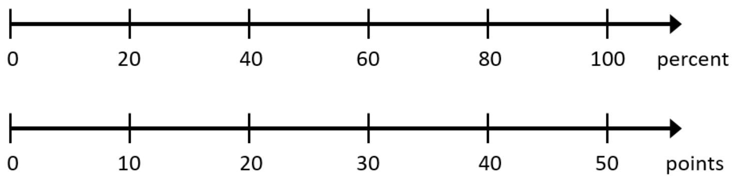
(4) $$A = \frac{P}{100} \times B .$$

Students will be expected to solve for the third unknown (part, whole, or percent), given the other two. The answer is going to involve either the multiplication of the two given numbers, or the division of one by the other. Keeping in mind that the goal of mathematics in the school curriculum is to develop students' ability to think through complex problems, it is not a good idea to provide students with three rules, depending upon which of A, P, B is the unknown. The following discussion, particularly the examples, are designed to stress understanding over automation.

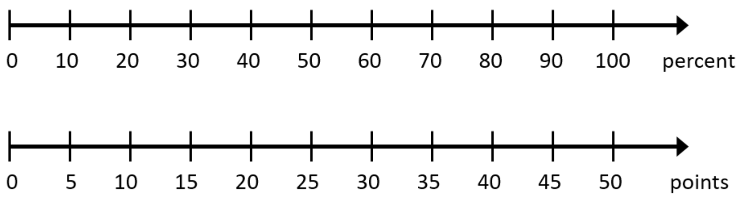
It is often useful to model the conversion of ratios to percentages by the *double number line*. Here, one number line (percent) is drawn and cut up into equal parts of 100 (often 25, 20, or even 10). Another number line is drawn parallel to the first and labeled so as to represent the whole of the quantity being discussed. In this way, the correspondence between value and percent can easily be seen. (Note: the number lines can be drawn in either order and cut up into friendly portions based on information given in the problem.) This may best be illustrated through an example.

Example 5: Danielle plays basketball. During the last game, she scored 10 of her team's 50 points. What percent of her team's points did she score?

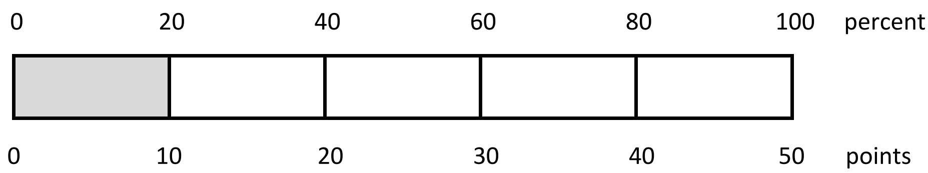
Solution. Since 10 is 1/5 of the 50 points scored, we could create a number line for points (cut up in groups of 10). Since the whole (50 points) was cut into 5 equal parts, we set 100 percent at 50 points and then cut that number line into 5 equal parts (1/5 of 100 = 20). Reading the model, we see that 10 points corresponds to 20% of the team's score.



Alternate solution: Since 5 is 1/10 of the 50 total points scored, the number line for points could have been split into 10 equal pieces. The percent number line would then also need to be cut into 10 equal parts, or groups of 10% each. Still we see that 10 points scored corresponds to 20% of the team's total points that game.



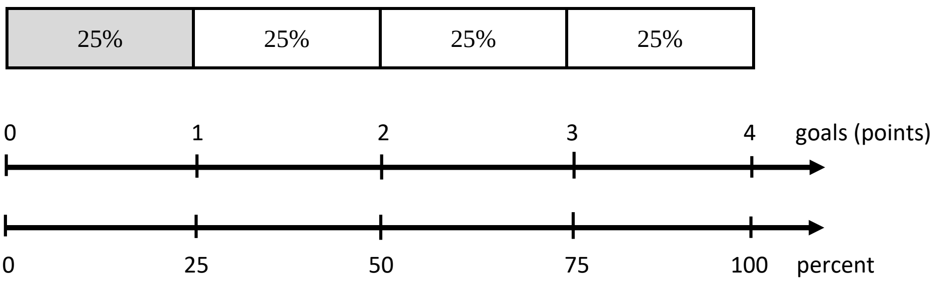
Notice how the double number line is similar to the bar (tape) model. If we use the top and bottom of the bar as our number lines, we can see the correspondence between points and percent.



One of the major uses of percentages is comparing parts from different wholes. This is illustrated in the next example.

Example 6: Miguel plays soccer. In the last game, he scored one of his team's four goals. What percent of his team's points did he score? How does his percent of points scored in the soccer game compare to Danielle's points scored in the basketball game? Who contributed more to the team's total score?

Solution. Students may be tempted to say that since Danielle scored 10 points and Miguel only scored one (goal), that Danielle scored more than Miguel. We would expect this as basketball games typically score many more points than soccer games. While it is true that Danielle scored more points (10 points vs 1), what we are looking for is the percent (portion) of the team's points scored by each player. A bar (tape) model or double number line show that Miguel scored 25% of his team's points. Since 25% is greater than 20%, Miguel scored a larger portion of the team's points with one goal (out of four) than Danielle did with 10 (out of 50) points.



Example 7. Let's work on understanding the percentage equation (4):

$$A = \frac{P}{100} \times B.$$

by doing a set of problems of the type described: find the value of the third variable, given the other two. The numbers have been chosen so that numerical computation is not an obstacle to just working with the definition. For example: 80 is 50% of what? 50% is equivalent to 1/2, so we can view the question as asking, 80 is half of what? Answer: 160.

A	*P*	*B*		*A*	*P*	*B*		*A*	*P*	*B*
20		400			32	100		12	100	
45		100			80	1000		12	10	
15		2000			20	50		12	500	
27		300			8	50		7	28	

Solution. First column. In this case we want to express the given ratio $A : B$ by a ratio whose second term is 400. So, for the first line, the ratio in question is 20:400. If we divide both terms by 4, we get 5:100, so 20 is 5% of 100. In the second line the ratio 45:100 is already in the desired form, so 45 is 45% of 100. In the third case, if we divide the second term by 20, we get 100; the first term now becomes 15/20 or 0.75. So, 15 is .75% of 2000. Finally, for the last line, divide by 3 to make $B = 100$ and find that 27 is 9% of 300.

Second column. Here we want to find *P*% of *B*. In the first line, *B is* 100, so we can say that 32 is 32% of 100. In the second line, 80 is 80% of 100, so, multiplying by 10, we have that 800 is 80% of 1000. As for the third line, we say, 20 is 20% of 100, so, dividing *A* and *B* by 2, 10 is 20% of 50. The last line is handled the same way, getting to the statement that 4 is 8% of 50.

Third column. We are given *A* and *P*, and we want to answer the question: *A* is *P* percent of what? So, the first line is easy: 12 is 100% of 12. As for the second line: if 12 is 100% of 12, then it is 10% of a number 10 times as large, namely , 20. Argue the third line the same way: if 12 is 100% of 12, then it is 500% of a number 5 times smaller than 12:, giving 2.4. The last line works the same way: start with 7 = 7% of 100, Now, if we multiply 7% by 4, we have to divide 100 by 4 to maintain equality. The result is 7 which is 28% of 25.

The table with the filled in answers is

A	P	B		A	P	B		A	P	B
20	5	400		32	32	100		12	100	12
45	45	100		800	80	1000		12	10	120
15	0.75	2000		10	20	50		12	500	2.4
27	9	300		4	8	50		7	28	25

To recapitulate (and express (4) in a different way): A is P% of B if the fraction A/B is equivalent to $P/100$. So, to say that A is 150% of B is to say that A is one and a half times as large as B, or "A is half again as large as B." Focusing on the increase of A over B, we might also say that "A is 50% *more than* B."

Let's illustrate this with the situation posed at the beginning of this subsection. As in some instances before, we will round decimal representations at two decimals. We have three quantities and we express their ratios as $B : A :: 4 : 5$, $C : A :: 6 : 5$. We then expressed these as fractions:

$$\frac{B}{A} = \frac{80}{100} \qquad \frac{C}{A} = \frac{83.33}{100} . \tag{5}$$

Since the denominators in (5) are the same, we can conclude that

$$B : C :: 80 : 83.3 , \quad \text{or as fractions} , \quad \frac{B}{C} = \frac{80}{83.33} = 0.96 = \frac{96}{100};$$

that is, B can carry 96% of what C can carry. Equivalently,

$$\frac{C}{B} = \frac{83.3}{80} = 1.04;$$

saying that C is 104% of B, and thus carries 4% more than does B.

Comparisons of Ratios, Fractions and Percentages

Another benefit of using models is that it helps students "see" the equivalence of different representations of the same value, whether it be a fraction, decimal, or percent. For example, 1/2 = 0.5 = 50%. Being able to move between different representations will be very helpful for students, especially as they encounter a variety of real-world problems. This even works with percentages over 100%. The point is: since we can rewrite ratios and fractions as percentages, we can use that to compare different ratios or fractions. Let us illustrate with another example:

Example 8. Find a fraction, decimal, and percent that represent the shaded portion of the image below. Note: there are two large rectangles, each representing one whole.

Solution. Since each whole is cut into four equal parts, the denominator is four. Because we have six shaded sections, the fraction is 6/4. This is equivalent to 3/2. We could also have started by noting that image shows one whole and two-fourths (or one half) of a whole. This can be represented by the following image:

We interpret this as 1 + 0.5 = 1.5. This decimal is equivalent to 150/100, or 150%.

In previous grades, students learned that a fraction represents one of a group of equal sized portions of a whole. As with fractions, each percentage must refer to some whole. It is important that the whole is clearly understood so confusion does not arise. In this manner, a percent is viewed as a part to total ratio. Often, we know the whole and the percent and are looking for the part (of the whole) that is represented by the given percent. This is illustrated in the next example.

Example 9. A company makes and sells candies that come in three colors: red, white, and blue. In each bag, 25% of candies are red, 50% are white, and 25% are blue. Phoebe bought a bag of candies. Make a tape diagram to illustrate this situation and then answer the following questions.

a). What is the ratio of red candies to total candies?

b). What is the ratio of red candies to white candies?

c). If the bag has 20 candies, how many are red? How many are white? How many are blue?

d). If the bag has 60 candies, how many are red? How many are white? How many are blue?

Solution. Since 25% is equivalent to 1/4 and 50% is equivalent to 1/2 = 2/4, we should split the tape into four equal pieces. Then one piece will represent red candies, two pieces will represent white candies, and the last piece will be for blue candies.

25% red	25% white	25% white	25% blue

a). From the tape diagram, we see that there is 1 red candy out of every 4 candies so the part to total ratio is 1:4.

b). The diagram also shows that for every red candy there are two white candies so the part to part ratio of red to white candies is 1:2.

c). Since the tape is split up into four equal parts, we divide the total number of candies, 20, into four equal parts of $20 \div 4 = 5$ candies each. We have included this on the diagram below. There are 5 red candies, $2 \cdot 5 = 10$ white candies, and 5 blue candies. Note that since the percent of red candies is the same as blue candies there will be the same number of red and blue candies.

25% 5 red	25% 5 white	25% 5 white	25% 5 blue

This problem could also have been solved with a partial table. 25% red candies means that 25 out of every 100 candies are red. Similarly 50 out of every 100 candies are white and 25 out of every 100 are blue. We note this in the second column of our table. Then, recognizing that 20 candies is one-fifth of 100 candies, we can take one-fifth of each of the numbers in the second column to find the number of candies of each color in the bag of 20 total candies (last column).

red candies	25	5
white candies	50	10
blue candies	25	5
total candies	100	20

d). As in the previous problem, we need to split the whole of 60 candies into four equal parts. This gives 60/4 = 15 candies to each box in the diagram. In this bag of candies, 15 are red, 15 + 15 = 30 are white, and 15 are blue.

25% 15 red	25% 15 white	25% 15 white	25% 15 blue

As above, this problem could also have been solved with a partial table. However, 60 is not a factor of 100. We need to do a little extra work. One method would be to scale up to 600 total candies (multiplying each value in the second column by 6) and then scale back to 60 candies by taking one-tenth of the values in the third column. This is shown in the table below.

red candies	25	150	15
white candies	50	300	30
blue candies	25	150	15
total candies	100	600	60

Another way to use a partial table to solve these types of problems is to scale 100 down to a factor of 60 and then scale up to the 60 total candies. One possible example would be to scale down to 20 [see part c).] and then scale up to 60. Yet another method would be to scale down to 10 and then up to 60. Here the intermediate steps involve parts of a candy but the end result is the same.

red candies	25	2.5	15
white candies	50	5	30
blue candies	25	2.5	15
total candies	100	10	60

Example 10. We return to the comparisons suggested in the introduction.

a. Which is larger: 5/8 or 3/5?

b. How much larger than 5/6 is 6/7?

c. On exam one I got 45 of 50 questions correct, and on the second exam, I got 68 of 75 questions right. Did I improve my performance?

d. In the first six months, the Ravson baby grew from 9 lbs to 14.5 lbs, and the Ikemba baby grew from 6.3 lbs to 10 lbs. Which baby has shown the greater growth?

e. At store A, they knocked $40 off the price of a $145 pair of shoes, and at store B, they have reduced their price of $160 for the same pair of shoes by 35%. Which is now the better price?

Solution. a). One thing we can do is to put these fractions over a common denominator. So, we have

$$\frac{5}{8} = \frac{25}{40} \qquad \frac{3}{5} = \frac{24}{40},$$

so 5/8 is larger than 3/5 (by 1/40). We can answer the question also by converting to percentages:

$$\frac{5}{8} = 62.5\% \qquad \frac{3}{5} = 60\% \ ,$$

from which we conclude that 5/8 is larger than 3/5 by 2.5%.

b). Reasoning in the same way with percentages:

$$\frac{5}{6} = 83.3\% \qquad \frac{6}{7} = 85.7\% \ ,$$

So, 6/7 is larger than 5/6 by 2.4%.

c). Here, 45/50 is 90% and 68/75 is 90.6%. Technically, we have to say that performance was improved by 0.6%, but that difference is so little, that it is best to conclude that you did almost the same.

d). This takes some thought. In this period the Ravson baby gained 5.5 lbs, and the Ikemba baby gained 3.7 lbs. So Ravson put on more pounds than Ikemba. But at the same time my gerbil went from 1/2 lb to 2 lbs, so only gained 1.5 lbs. But my gerbil is now 4 times as large as he was 6 months ago, and neither the Ravson nor the Ikemba family can make *that* claim. The resolution is this: if we want to make a fair comparison we should compare the ratios

$$\text{weight today} \ : \ \text{weight six months ago} \ .$$

Expressing this ratio as a percentage we have that the comparison of "weight today" to "weight 6 months ago" is:

$$\text{Ravson} : \frac{14.5}{9} = 161\% \qquad \text{Ikemba} : \frac{10}{6.3} = 159\% \qquad \text{Gerbil} : \frac{2}{1/2} = 400\% \ .$$

We conclude that the Ravson baby grew slightly more than the Ikemba baby, but both were slight compared to the growth of the gerbil.

e). We have to compare the new prices. The new price at store A is \$105, and the new price at store B needs a little more thinking. The original price of \$160 has been reduced by $0.35 \times \$160 = \$56$. So the new price at store B is $\$160 - \$56 = \$104$. The new price at store B is \$1 better than that at store A, so there really isn't much difference. We might ask: "Which store made the larger reduction?" but the answer is not relevant to the choice of store for this purchase.

The value of percentage considerations in the above problem is in assessing the question of *how big* is the difference between two numbers. The next example illustrates its value in comparing a set of numbers.

Example 11. List these numbers in increasing order:

7/9 2/3 65% 0.62 7/10

SOLUTION. First of all, one may perceive 65% as a percentage, not as a number. But a percentage *is* a number: 65% literally means 65 per hundred, which is 65/100. After clarifying that issue (if it arises), the way to make the comparison is to convert each number so that they are all represented in the same way. We can convert to fractions, decimals or percentages. Converting to fractions is not advised, as it leads to more arithmetic (which is larger: 65/100 or 7/10?). The alternatives are decimals or percentages (which are the same, since 83% = 0.83), So, let's convert to decimals. We find, up to two places:

7/9 = 0.78 , 2/3 = 0.67 , 65% = 0.65 , 0.62 7/10 = 0.70

and so, the ordering is

$$0.62 < 0.65 < 0.67 < 0.70 < 0,78.$$

Real World Problems

Example 12. The economy of the island nation of Gonzovilla depends on its immigrant population. Therefore, the government has decided to try to maintain their immigrant population at 35%. At present there are 480,000 native Gonzovillans on the island. At what immigrant population level should the government aim?

SOLUTION. If the immigrant population is to be maintained at 35%, then the percentage of natives (non-immigrants) is 65% of the total population. So we have the equation

$$480,000 = 0.65 \times (\text{total population}) ,$$

telling us that the total population is $480,000 \div 0.65 = 738,462$. The immigrant population will be capped at 35% of that number, which is $0.35 \times (738.462) = 258,462$.

There is another way to solve this, and that is by transforming the given information to a part to part ratio. That is, the immigrant population should be to the native population as 35:65. Furthermore, the native population is 480,000. Letting IP represent the immigrant population, we get the ratio $IP : 480,000 :: 35 : 65$, leading to the equation

$$\frac{IP}{480,000} = \frac{35}{65}.$$

and so $IP = (35/65)(480000) = 258462$.

Example 13. Lorelle, Smitty and Daphne went into the brush behind the school to pick blueberries. Lorelle is 40% faster than Smitty in picking blueberries, and Daphne can pick 5 cups of berries in the time Smitty picks 4 cups. One afternoon, Smitty came home with a quart and a half of berries. How many quarts did Lorelle and Daphne have?

SOLUTION. The information given to us is that, in the same time period, Lorelle will pick 140% of what Smitty picks, and Daphne will pick 5/4 as much as Smitty picks. Converting 5/4 to a percentage: Daphne picks 125% of what Smitty picks.

On the particular afternoon in question Smitty came home with a quart and a half of berries. This tells us that, on that same afternoon,

$$\text{Lorelle picked } 1.4 \times 1.5 = 2.1 \text{ quarts of berries,}$$

and

$$\text{Daphne picked } 1.25 \times 1.5 = 1.875 \text{ quarts of berries.}$$

As we saw in Example 7, percent problems generally involve three components: a part, a whole, and a percent. There are three common types of percent problems, finding the missing component (part, whole, or percent) given the other two. In the case of the basketball and soccer examples, Examples 5 and 6, we knew the part (number of points/goals scored by the child), the whole (the total number of points/goals scored by the team), and were trying to find the percent contributed by that one player. In the candies problem (Example 9), we knew the percent (percent of candies of each color) and the whole (how many candies in a bag), and were looking for the part (the number of candies of each color in the bag).

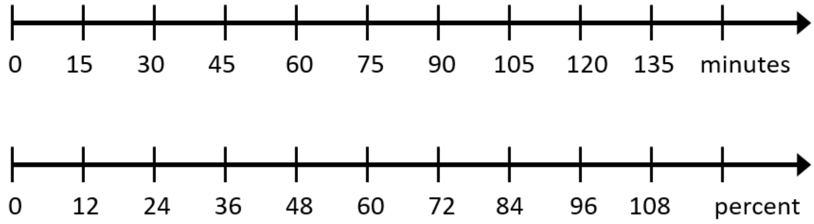
The third type of problem gives the part and its corresponding percent but asks us to find the whole. This is illustrated in the following problem.

Example 14. The battery on Oliver's mp3 player is completely exhausted. He plugs it in and it begins to recharge. Fifteen minutes later, he finds that the battery is at 12%. At this rate, how long does it take to charge the battery completely (from empty)? First, estimate your answer and then find the exact value.

SOLUTION. First, let us estimate. Since 15 and 12 are relatively close, and 15 is a little bigger than 12, we would estimate that the mp3 player will take a little more than 100 minutes for the battery to reach 100% charge.

Several different methods could be used to solve this problem. Two will be shown below: a) double number line, b) partial table.

a). We can set up the double number line and iterate the intervals for 15 minutes and 12%.



Note that since 12 is not a factor of 100, in this problem, 100% does not appear as one of the iterations. However, it is between 96% (120 minutes) and 108% (135 minutes). In fact, to get from 96% to 100%, one needs to add 4% to 96%. Since 4% is one-third of a group of 12%, we need to use one-third of a group of 15 minutes: ($1/3 \cdot 15 = 5$ minutes). Therefore, the total amount of time to charge the battery is 120 minutes plus 5 minutes = 125 minutes. This agrees with our estimate of a bit over 100 minutes.

b). Since 12 is not a factor of 100, we need to change the scale so that it is. One easy way would be to take one-third of 12 to get 4, which is a factor of 100 (third column). We then multiply the third column by 25 to get to 100. Again, our result is the same, 125 minutes, and it agrees with our estimate.

percent	12	4	100
minutes	15	5	125

Example 15. At our school, 8% of the students live within walking distance. 4 out of 5 students are not within walking distance, but are on school bus routes. The remaining 45 students come to school via other means of transportation. Fill in the missing entries in this table.

Sector	Number	Percentage	Ratio
Walk to School		8%	
Bus to School			4 out of 5
Other	45		

Solution. It seems that we have not been given enough information to solve the problem. But let's first fill in what we can. In the first line we can convert the 8% to a ratio of 8:100, which reduces to 2:25. And in the second line, we can convert the ratio "4 to 5" to the percentage 80%. That tells us that 8% + 80% = 88% of the students either walk to school, or bus to school. That means that the remainder (Other) is 12%, which is equivalent to the ratio of 3 out of 25, leading to the new table:

Sector	Number	Percentage	Ratio
Walk to School		8%	2 out of 25
Bus to School		80%	4 out of 5
Other	45	12%	3 out of 25

In particular, we now know that 45 is to the total school population as 3 is to 25, thus (since $45 = 15 \times 3$), the shool population is $15 \times 25 = 375$. Now that we know the full population, 80% of them take the bus, and that is $0.8 \times 375 = 300$, and 8%, or $0.08 \times 375 = 30$ walk to school. Here is the completed table:

Sector	Number	Percentage	Ratio
Walk to School	30	8%	2 to 25
Bus to School	300	80%	4 out of 5
Other	45	12%	3 out of 25

We can make a quick check of our calculations: by totaling the figures in each column we should get the row

Total	375	100%	25 out of 25

To sum the entries in the last row, we have to change "4 out of 5" to the equivalent "20 out of 25."

Example 16. In our class, our final grade will be calculated on the basis of five examinations, all of the same weight. My grades on the first four exams were: 85%, 95% , 75% and 90%. If I do as well as I did on my best exam, what will be my final grade?

Solution. We aren't told how many problems were on each exam, but we were told that they are of equal weight. So, it is as if we got 85 out of 100 problems correct on the first exam, 95 out of a hundred correct on the second exam, and so forth. In total, then, we have gotten $85 + 95 + 75 + 90 = 345$ problems correct out of 400. If I do as well on the fifth as my previous best, that will be 95 out of 100 correct, and in total, I'll have scored $345 + 95 = 440$ out of 500 correct. That is the same as 88 out of a hundred, so my final score will be 88%.

Example 17. Wallace has a summer job that he will keep for five more years. He is paid at the rate of $12.00/hour. Supposing that each summer, he gets a 10% raise, what will be his hourly salary in the fifth year?

Solution. One might like to argue: since he gets a 10% raise each year, in total his salary is raised by 50%, so he will be earning $12.00+ $6.00 = $18.00 per hour. However, Wallace's annual increase is 10% of the salary of the preceding year, and *not* that of the first year. So, we have to calculate Wallace's hourly pay year by year: each number is 10% greater than the preceding one. Otherwise put, for each entry Year $k + 1 = 1.1\times$ Year k.

This Year	Year 1	Year 2	Year 3	Year 4	Year 5
12.00	13.20	14.52	15.97	17.57	19.33

Section 2. Division of Fractions

Apply and extend previous understandings of multiplication and division to divide fractions by fractions. 6.NS

Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ *and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that* $(2/3) \div (3/4) = 8/9$ *because 3/4 of 8/9 is 2/3. (In general,* $(a/b) \div (c/d) = ad/bc$*.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi? 6.NS.1*

Let us recall the development of the operations of multiplication and division. Multiplication is introduced as repeated addition: 5×6 is the number to be associated to a grouping that brings together 5 copies of a 6 element group. This is understood to be the same number as grouping together 6 copies of a 5 element group (this is the commutative property of multiplication: $a \times b = b \times a$). Subsequently, division is introduced as the inverse operation: $12 \div 3$ is the number to be associated to each piece of the group of 12 when it is partitioned into 3 equal pieces. Alternatively, $12 \div 3$ is the number of groups we can make when each group gets 3 units. In summary, $A \times B$ is the number of elements in the resulting group when we bring together A copies of B, and $A \div B$ is the number of elements in each resulting group when we partition A into B pieces of equal size. This is the *partative* meaning of division; there is also the *quotative* meaning: $A \div B$ is the number of groups that can be made from A elements where there are B elements per group. Then the arithmetic becomes complicated, but the model persists. What is $12 \div 5$? How much do we have in each new group when we divide 7 into 4 equal pieces? Fractions respond to the need. For a whole number q, $1/q$ represents the typical piece that results when we divide 1 into q pieces, all of the same size. Then p/q represents p copies of $1/q$: that is $p \times (1/q)$.

Now, what is $(1/q) \times p$? Keeping the model consistent, it is viewed as $1/q$ copies of p, that is, a measure of the size of one piece, when p is divided into q pieces of equal size. At this time it should be noted that $(1/q) \times p$ is the same as $p \times (1/q)$ - that is the size of the $1/q$th copy of p is the same as the size of p copies of the qth part of 1.

These definitions extend the basic rules of arithmetic to all fractions:

For whole numbers p and q

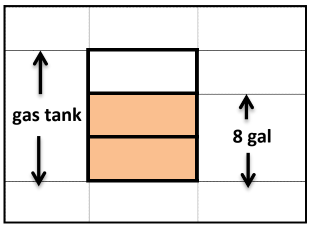
$$p \times q = q \times p\,, \quad \text{for } q \neq 0\,, \frac{1}{q} \times p = p \times \frac{1}{q}\,, \quad p \div q = \frac{1}{q} \times p$$

and $\div$ undoes $\times$: that is, if $a = b \times c$ for nonzero numbers a, b, c, then $c = a \div b$ and $b = a \div c$.

We now show that these rules apply for all nonzero fractions as well. For that, we need to have an understanding of what is meant by division by a fraction.

Example 18. When I received my new car, it had almost no fuel in it, so I brought it to a gas station and put 8 gallons in it. The fuel meter now registered 2/3 full. What is the full capacity of the gas tank?

Solution. The bar diagram to the right illustrates this problem. To solve it, I have to find out how much gas is in one of the orange bars. Then, since the full tank comprises three such bars, the capacity of the full tank is 3 times that number. Now, I know the contents of the two orange bars is 8 gallons, so one bar contains $8 \div 2 = 4$ gallons. Finally, 3 bars, the full tank, comprises $3 \times 4 = 12$ gallons.



Let's analyze this solution. The question is this: if 2/3 of {full tank} is 8 gallons; what is {full tank}? We know that a full tank has to be more than 8 gallons, and we know that we have to either multiply 8 by 2/3 or 3/2. So, to get a large number, we better go with $8 \times (3/2) = 12$ gallons.

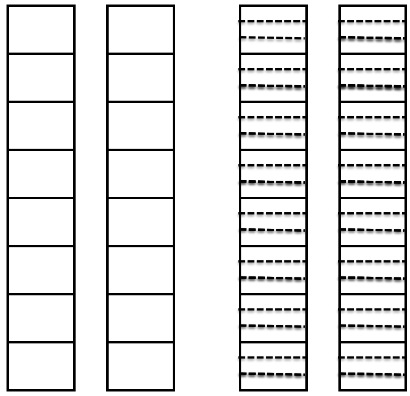
That's a good "best guess argument," but here is a rational explanation. We first divided by 2 to find out how much is in 1/3 of {full tank}, and then multiplied by 3 to get the capacity of the full tank. In symbols, this argument is:

$$\text{the solution of } \frac{2}{3}X = 8 \quad \text{is } X = 3 \times (\frac{8}{2})\,, \text{or } \; X = \frac{3}{2} \times 8 = 12\,.$$

Now, to be arithmetically consistent, if we recall that the solution to the equation $aX = b$ is $X = b \div a$, we should have $X = 8 \div \frac{2}{3}$. This confirms for us that the way to divide by a fraction $\frac{p}{q}$ is to multiply by $\frac{q}{p}$:

$$A \div \frac{p}{q} = A \times \frac{q}{p}\,.$$

Example 19. I ordered 2 pizzas for our group meeting, and each pizza comes divided into 8 equal slices. That's good if no more than 16 people attend our meeting, but it is likely more will come. Suppose I cut each slice into thirds, and - assuming that each person will eat 2 such pieces - how many guests can I have?



Solution. I am asking: how many 2/3-slice portions are there in 16. Put another way, if I divide 16 slices into 2/3-slice portions, how many portions are there? The answer is

$$16 \div \frac{2}{3} = 16 \times \frac{3}{2} = 24 ,$$

if we go by the rule that division by p/q is the same as multiplying by q/p.

Let's see what a bar diagram does for us. On the left side we have modeled the two pizzas as presented, each cut into 8 slices. On the right hand side we see the two pizzas with each original slice cut into thirds. On the right hand side, there are 48 one-third slices: under the assumption that each participant eats two slices, we can have $48 \div 2 = 24$ participants.

Let's follow the mathematics of this representation. In the diagram on the left, we have 16 pizza slices. Now, we divide each slice into thirds: that number is $16 \div \frac{1}{3} = 48$. Now, if each participant eats two slices, the number of participants is 24. Following the arithmetic:

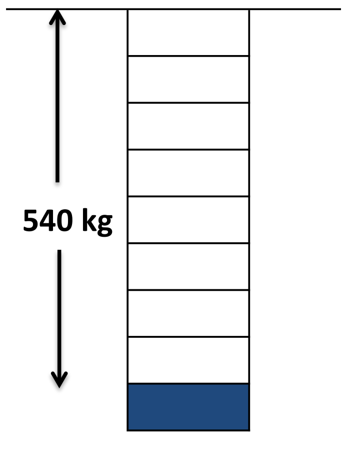
$$16 \div \frac{1}{3} = 16 \times 3 = 48 \quad \text{and so } 16 \div \frac{2}{3} = (16 \times 3) \div 2 = 24.$$

Example 20. When I had a construction job, the foreman asked me to dig a hole of diameter 3 feet. I asked "how deep should it be?" and his reply was that he would let me know when he came back. When he returned, I had already removed 540 kg of earth. He measured the depth of the whole and told me that I have an eighth more earth to remove.

a. According to the foreman, what percentage of the desired hole had I already dug?

b. In weight, how much earth will I have removed in total?

c. If a cylinder of earth of diameter 3 feet and depth 1 foot weighs 75 kg, how deep is the hole when the job is completed?



Solution. To the right is a bar diagram depicting the situation as described. Since the foreman said that I have another eighth to go, I have divided the existing hole into eights, and then added another eighth, filled with earth, that I have yet to remove.

a. Using this graphic, a unit consists of one of the boxes. I have dug 8 boxes, and have one more to go, so 8/9 of the job is done. In percentages: 8/9 = 89%.

b. The work I've done so far is 8/9 of the full job, and has excavated 540 kg of earth. That is, 540 kg is 8/9 of the weight of the full job, so to find the weight of the full job I have to calculate

$$540 \div \frac{8}{9} = 540 \times \frac{9}{8} = 607.5$$

kilos. We could also have calculated this using the bar diagram. 8 units (of hole) weighs 540 kg, so one unit is $540 \div 8 = 67.5$ kg. When the job is done, I will have excavated 9 units, which in weight is $9 \times 67.5 = 607.5$ kg.

c. The question is: if one foot of my hole weighs 75 kg, how deep is the hole? We can answer this, using the bar diagram. One box weighs 67.5 kg. One foot weighs 75 kg. So the fraction, in feet, one box is $67.5/75 = 0.9$ feet deep. The finished hole comprises nine boxes, so is $9 \times 0.9 = 8.1$ feet deep. Alternatively, we can say that we have dug 607.5 kg of earth, and one foot comprises 75 kg of earth. So in feet, we have dug $607.5 \div 75 = 8.1$ feet.

Our examples up to now have been with fractions p/q with $p < q$, but these rules holds for all p and q so long as $q > 0$.

Section 3. Ratio Reasoning and Measurement Conversion

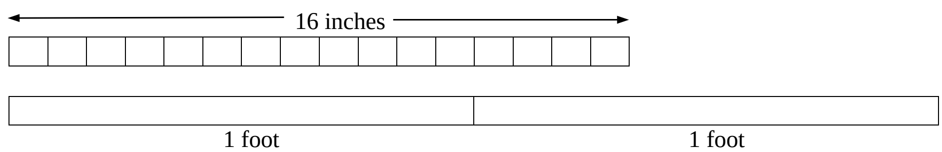
Use ratio reasoning to convert measurement units; manipulate and transform units appropriately when multiplying or dividing quantities. 6.RP.3d

Often problems of measurement involve more than one way of measuring; in order to resolve the problems, a conversion of units is necessary. In this section we consider several contexts in which this situation arises, both to familiarize students with the concept of conversion, and also the relationships among common units.

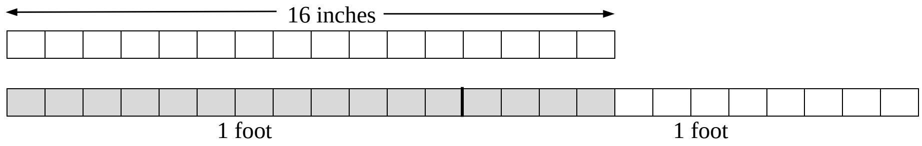
As they progress working with fractions, students should notice patterns in the multiplication and division. For example, if we multiply some number, a, by a value, b greater than 1, we are increasing the amount of the quantity. On the other hand, when we multiply our number by a value, b between 0 and 1, we decrease the quantity. As examples, letting our chosen number $a = 6$, then if $b = 1.5, b \cdot a = 1.5 \cdot 6 = 9$ (bigger than 6) but if $b = 0.3$, then $b \cdot a = 0.3 \cdot 6 = 1.8$ (smaller than 6). In fact, the larger the value we multiply by (greater than 1), the bigger the product will be. Consider $2.5 \cdot 6 = 15$, which is bigger than $1.5 \cdot 6 = 9$ because we are taking more groups of 6 (in fact one more group of 6 which is why the answer to $2.5 \cdot 6$ is exactly $15 - 9 = 6$ larger than the result of $1.5 \cdot 6$). Also, the smaller the (positive) value we multiply by (between 0 and 1), the smaller the product will be. If we multiply 6 by 0.2 we get 1.2, while multiplying 6 by 0.01gives 0.06. Of course, the tipping point for multiplication is when you multiply by 1: $1 \cdot a = a$ for every number a. Multiplying by 1 does not make the result bigger or smaller; because of this important property, 1 is called the "multiplicative identity."

After their work in the previous section, students should notice that if you divide a number, a, by a value greater than 1, the quotient will be smaller than the dividend. An example is $6 \div 1.5 = 4$: the quotient 4 is smaller than the dividend of 6. Students may think of this problem as "How many groups of 1.5 units are there in 6 units?" The answer is less than 6 because it takes more than 1 unit to make a group. If the value of the divisor increases, the quotient will get smaller, as in $6 \div 5 = 1.2$. However, if you divide by a value between 0 and 1, the quotient will be bigger than the dividend. To illustrate: $6 \div 0.5 = 12$: here the quotient 12 is larger than the dividend 6. Students may think of this problem as "How many groups of one-half (of a unit) are in 6 units?" The answer is greater than 6 because each portion is less than a unit so we can make more than 6 groups. As the (positive) divisor gets closer to 0, we see that the quotient gets larger: $6 \div 0.2 = 30, 6 \div 0.1 = 60, 6 \div 0.05 = 120$, and so forth.

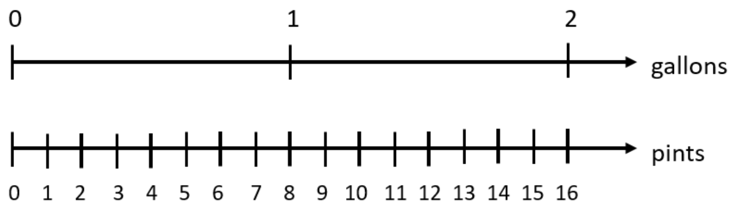
This number sense comes into play when doing conversions of measurement. Recognizing whether you are going from a larger unit of measure to a smaller (or vice versa) can help a student recognize whether the solution method and answer are reasonable. If a student is converting from inches (small unit of measure) to feet (larger unit of measure), then the quotient (answer) should be a value smaller than the original number. For example, a student may be converting 16 inches to feet. Since feet are larger units of measure, the number (of feet) will have to be smaller than 16. This is illustrated in the following diagram.



From the image, we see that 16 inches is more than 1 foot but less than 2 feet (and both of these numbers are smaller than 16). We can cut each of the feet into 12 inches to find that 16 inches is 16/12 = 4/3 = 1 1/3 feet, as shown below.



On the other hand, if a student is converting from gallons (larger unit of measure) to pints (smaller unit of measure), then the quotient will be a value larger than the dividend. If a pitcher holds 2 gallons of punch and each student gets 1 pint of punch, how many students can we serve? Since a pint is much smaller than a gallon, we expect the result to be larger than 2. A double number line could be used to visualize this. Recalling that there are four quarts in a gallon and two pints in each quart, we get the following.



This shows that 2 gallons is equivalent to 16 pints (and 16 is a larger value than 2 because pints are a smaller unit of measure than gallons).

Now, one fact with which we have to reckon is the fact that there are two systems of measurement: that formerly known as the British system, and the metric system. In the United States, we are familiar with, and commonly use the British system, to which we will refer as the *customary* system. In the customary system, words that are used to describe distance, weight and volume are rooted in the basic measures people, thousands of years ago, wanted to make. A *foot* is literally an estimate of the average length of a human foot. An *inch* is, roughly speaking, the length of the tip of one's thumb to the first knuckle. A *yard* is the length of an adult human running stride. The word *mile* comes from the Latin *mille* meaning a thousand. So, a mile was meant to mean a thousand paces (while walking from here to there). Information like this can be found at www.mentalfloss.com.

The use of different words comes about because we need language suitable for every scale. If we want to measure the length of our newborn, we'll say something like 18.5 inches; but we certainly don't want to measure the distance from our house to the grocery store in inches. For that, we're interested in how many paces, or yards, are involved. Now the distance from our school to Chicago will be measured in thousands of paces (miles).

In the United Kingdom, these measures were used widely in the dark ages, but were not standardized. The number of inches in the height of a wall was very much dependent upon the size of the builder's thumb; the distance across town on the size of one's stride. This explains a strange feature of early medieval buildings: parallel walls are not really parallel and in no way congruent - but close enough to keep the building from collapsing. One of the accomplishments of the early renaissance was the standardization of measurements throughout Europe. So, when we got yardsticks, the size of the inch was the same for all of us.

Now, metric units came about in an interesting way. Until the end of the eighteenth century, just about every language in Europe had its own system of measurement; the French had *pouces* and *libres*, the Italians had *police* and *etti* and so forth. But the revolution in France at the end of the eighteenth century changed everything: the form of government, the calendar and systems of measurement. The idea was to have a system of measures where the change from one unit of measurement was always a factor of 10. The way measures of different properties were related was through water. Once a measure for the meter was established, we could talk about thousands of meters (kilometers) and thousandths of meters (millimeters). Now one cubic centimeter (abbreviated as cc) was a cube of side length one centimeter on each side. It was decreed that when filled with water, this was one gram of weight and one milliliter of volume. So, now and forever after, one cubic centimeter was equivalent to one gram and one milliliter, if what is being measured is water.

When Napoleon became emperor of most of Europe, the metric system was established everywhere. This lasted 7 to 12 years, depending upon how long one's community was Napoleonic. But on the defeat of Napoleon in 1812, all of Europe reverted to their own local system of measurement. However, this was also the time of the Industrial Revolution and the internationalization of commerce, so the business community could not help but appreciate the great advantage achieved by a uniform system of measurements. As it turns out, by the time of the revolutions of 1848, all of continental Europe had accepted the metric system of measurement. Great Britain was never part of the Napoleonic empire, and so held on to the medieval measurements until recently. As the United States was first established as a British colony, the British system was inherited, became the customary system, and remains so today.

Students, used to the customary system of measurement, have an intuitive feel for the relations among units of measurement of distance and of weight, but less so of measures of volume. However, conversion to the metric system presents obstacles, partly because of unfamiliarity with the system, and partly because the conversion numbers are not whole numbers.

First, let us provide a table of measurement conversions which will be used in these materials.

Table 1. Conversions of Measures

Distance			**Volume**			**Weight**	
1 foot	12 inches		1 tbsp	3 tsp		1 pound	16 ounces
1 yard	3 feet		1 cup	16 tbsp		1 ton	2000 lbs
1 mile	1760 yards		1 pint	2 cups		1 pound	0.454 kg
1 inch	2.54 cm		1 quart	2 pints		**Time**	
1 meter	3.28 feet		1 gallon	4 quarts		1 minute	60 seconds
1 yard	0.914 m		1 liter	1.057 qts		1 hour	60 minutes
1 meter	1000 mm		1 liter	1000 cc		1 day	24 hours

A partial table is another way that students can convert measurements (and verify that their solution is reasonable). This is demonstrated in the next example.

Example 22. If a student is 140 cm tall, how many meters is he/she?

Solution. #1: There are 100 centimeters in 1 meter so the answer, in meters, should be smaller than 140.

	centimeters	meters	
	100	1	
× 1.4	140	1.4	× 1.4

Solution. #2: If the student doesn't realize that 140 is 1.4 groups of 100, then the student could iterate 100 down to a factor of 140 (like 20) and then iterate up to 140. This is shown in the table below.

	centimeters	meters	
	100	1	
$\div 5$	20	0.2	$\div 5$
$\times 7$	140	1.4	$\times 7$

The example above converted within the same measurement system. Sometimes we need to convert between systems. For example, we might want to find out how many inches tall the 140 cm tall student is. Line 4 of Table 1 above tells us that there are 2.54 cm for every 1 inch. Since inches are larger units of measure than cm, the value should be smaller than 140 cm. A table can help us find the answer.

	centimeters	inches	
	2.54	1	
$\div 2.54$	1	$\frac{1}{2.54} = 0.394$	$\div 2.54$
$\times 140$	140	55	$\times 140$

Rounded to the nearest inch, 140 cm is 55 inches. Note that the first row of the table was the unit rate of centimeters per inch. Since 2.54 is not a factor of 140, we scale down to something easy, 1 cm, by dividing by 2.54. This actually gives us the unit rate of 1/2.54 = 0.394 inches per centimeter. Finally, we scale up to the desired number of centimeters by multiplying by 140, getting 55 inches (rounded to the nearest inch).

Example 23: The traveler's suitcase weighs 46 pounds. The airline charges extra for luggage that weighs more than 25 kilograms. How much does the traveler's suitcase weigh? Will she have to pay extra?

Solution. Line 3 on the right side of Table 1 tells us that there are 0.454 kilograms in one pound. Thus, since kilograms are the larger unit of measure, we should expect a number smaller than 46 for the weight in pounds. Is it less than 25? Calculate:

$$(0.454 \ \text{kg per pound}) \times (46 \ \text{pounds}) = 20.88 \ \text{kg}.$$

Since this is less than 25 kg, she will not have to pay extra.

Example 24. a). Rodrigo can run one kilometer in 5 minutes. In the same 5 minutes, Miroslava runs 7/10 of a mile. Which runner is faster - and by how much?

b) Now Emilio runs 1 mile in 8 minutes and Usana runs 100 meters in 10 seconds. Of the four runners, who is the fastest.

Solution. a) We have to change the distances to the same unit of measure. According to our table, 1 mile is 1760 yards, one meter is 3.28 feet, and one yard is 3 feet. Let's change everything to yards. 7/10 of a mile is $(7/10) \times 1760$ yards, or 1,232 yards. Since a kilometer is 1000 meters it is $1000 \times 3.28 = 3280$ feet. Converting feet to yards, we get $(1/3) \times 3280 = 1,093.3$ yards. In 5 minutes Miroslava has run about 140 yards further than Rodrigo.

b) Now we have to change both the distance and time units to be the same. Let's convert to yards in distance, and to 5 minutes in time. Emilio runs 1 mile in 8 minutes, and 5/8 of 8 minutes is 5 minutes. Emilio runs 5/8 mile in 5 minutes, In yards this is $5/8 \times 1760 = 1,100$ yards. Now, for Usana, 6×10 seconds is a minute, so 5 minutes is $5 \times 6 = 30$ 10-second intervals. If Usana can keep up the same pace, he will run $30 \times 100 = 3000$ meters in 5 minutes. 3000 meters is $3000 \times 1.094 = 3,282$ yards, so he theoretically can run 3,282 yards. The distances (in yards) run in 5 minutes is shown in the table in decreasing order:

Usana	**Miroslava**	**Emilio**	**Rodrigo**
3282	1232	1100	1093

So, Usana is the fastest, and the others are more or less bunched together. This problems shows the futility of comparing sprinters with long distance runners; bursting your heart for 10 seconds is not something that can be maintained for 30 10-second stretches one after the other.

Example 25. My local grocer sells imported olive oil by the liter, and domestic olive oil by the quart and gallon. Here are the brands, amounts and prices:

Esplanado	**Ixtaca**	**Corofino**	**Napa**	**J.S. Orchards**
3 liters	1 liter	5 liters	1 gallon	1 pint
$25	$9	$42	$35	$6

Order the vendors according to value.

SOLUTION. We really can't do this with the given data, because we have not quantified quality. One oil may come from small mountain top villages in a remote place with perfect climate, and another may come from an industrial orchard on a reclaimed part of a searing desert. But since we don't have that information, the best we can do is compare quantity of oil per dollar. Looking at the information, and our table of equivalences, it makes sense to change to quarts as the common unit, because our table allows us to convert directly to quarts. We want to end up with this statement for each brand: 1 quart costs $ . . . .

Esplanado: We are given the ratio 3 liters : $25. We want a 1 in the quantity position, so we divide by 3 to get 1 liter : $8.33. But one liter is equal to 1.057 quarts, so we can rewrite this ratio as 1.057 quarts : $8.33. Now, divide by 1.057 to get 1 quart : $7.88.

Ixtaca: We start with 1 liter : $9.00 and, as above, convert to quarts: 1.057 quart : $9.00. Again, divide by 1.057 to get 1 quart : $8.51.

Corofino: Use the same process: Start with 5 liters : $42, go to 1 liter : $8.40 and then its equivalent 1.057 quarts : $8.40, and finally divide by 1.057 to get 1 quart : $7.95.

Napa: 1 gallon is 4 quarts, so this gives us the ratio 4 quarts : $35.00. Divide by 4 to get 1 quart: $8.75.

J.S. Orchards: 1 pint costs $6, so 1 quart (= 2 pints) costs $12.

The cost per quart for each brand is displayed in this table:

Esplanado	Ixtaca	Corofino	Napa	J.S. Orchards
$7.88	$8.51	$7.95	$8.75	$12.00

There are several other ways of doing these computations. Here's one:

Esplanado: The information is given in terms of liters, and we want quarts. So, start with the conversion 1 liter : 1.057 quarts. Since the cost given is for 3 liters, multiply by 3: 3 liters : 3.171 quarts. So now we know that 3.171 quarts costs $25. To find the cost of one quart, divide $25 by 3.171 to get $7.88.

Given the fact that there are several ways of doing conversion problems, students may ask, ”When do I multiply and when do I divide?” This is a good question, because it illustrates the significance of understanding the development of a method of solution. Then, in executing each step, the answer to the question is obvious. It is never obvious if one has just memorized rules.

Example 26. I sit in my math class for 1.5 hours in each of three days every week. A semester consists of 14 weeks. In total, in terms of days, how much time do I spend in math class each semester?

Solution. Let's first make a total in terms of hours. Each week, I spend $3 \times 1.5 = 4.5$ hours in math class. So, in a total of 14 weeks, I have spent $14 \times 4.5 = 63$ hours in my math class. That is $63/24 = 2.625$ days in math - each semester.

What we here are calling *units* are referred to by physical sciences as *dimensions*, a word to be taken in its broadest sense: meaning some way of measuring a physical attribute, such as weight, time, temperature, volume, or distance. Since the laws of physics can be stated as providing relationships among these attributes, it is essential that the scientist have a great facility in making conversions among these dimensions. The use of this facility is called *dimensional analysis*. This subject goes beyond the content level of grade 6 mathematics, but because of its immense importance, we provide here an introduction. The tools of dimensional analysis are the ratios among the dimensions involved in the experiment, such as those listed in Table 1. But, even more is involved: if the experiment has to do with falling objects or frictionless sliding, one of the attributes to be considered is speed, meaning the ratio of distance to time.

Dimensional analysis is based on four mathematical relations

a. For any nonzero number n, $\frac{n}{n} = 1$.

b. Any number, n, multiplied by 1 is still n : $n \cdot 1 = n$.

c. For any positive number n,

$$\frac{a}{b} = \frac{a \cdot n}{b \cdot n} .$$

d. For any two nonzero numbers a, b,

$$\frac{a}{b}\frac{b}{a} = 1 .$$

These properties can be explained in terms of ratios, since the fraction a/b represents the ratio $a : b$.

a. If two quantities are in the ratio 1:1, we mean that the numbers of these quantities are the same.

b. This follows from a. by multiplying both sides by n.

c. This says that the ratio $a : b$ is equivalent to the ratio $na : nb$ for any positive number n.

d. This says that given two quantities A and B, then if A is to B as a is to b, then B is to A as b is to a. For example, if we say “in our school, there are 5 girls for every 4 boys,” that is the same as saying “in our school, there are 4 boys for every 5 girls.”

Dimensional analysis applications can best be understood through examples.

Example 27: Use dimensional analysis to convert:

a. 16 inches to feet.

b. 7 feet to inches.

c. 140 cm to km.

d. 46 lbs to kg.

e. 25 kg to lbs.

Solution. a. There are 12 inches to a foot, so we can say that the quotient

$$\frac{\text{1 foot}}{\text{12 inches}}$$

is equal to 1. Then we calculate:

$$16 \text{ inches} \times \frac{\text{1 foot}}{\text{12 inches}} = \frac{16}{12} \text{ feet} = 1.33 \text{ feet.}$$

b. There are 12 inches to a foot, so we can say that the quotient (12 inches)/(1 foot) is equal to 1. Then we calculate:

$$7 \text{ feet} \times \frac{\text{12 inches}}{\text{1 foot}} = 7 \times 12 \text{ inches} = 84 \text{ inches.}$$

c. There are 100 centimeters to a meter. Thus

$$140 \text{ cm} \times \frac{\text{1 meter}}{\text{100 cm}} = \frac{140}{100} \text{ meter} = 1.4 \text{ m.}$$

d. There are 0.454 kg to a pound. So:

$$46 \text{ lbs} \times \frac{\text{0.454 kg}}{\text{1 lb}} = 46 \times 0.454 \text{ kg} = 20.88 \text{ kg.}$$

e. There are 0.454 kg to a pound. So:

$$25 \text{ kg} \times \frac{\text{1 lb}}{\text{0.454 kg}} = \frac{25}{0.454} \text{ lb} = 55.07 \text{ lb.}$$

Example 28: Example: Paulo ran 800 meters in 3 minutes. Express his running ability in terms of miles per hour.

Solution. Start with the given information and multiply by unit fractions so that the unwanted units of measure divide out. Keep working until you arrive at the desired units, miles per hour (mph), in this case.

$$\frac{\text{800 m}}{\text{3 min}} \times \frac{\text{1 km}}{\text{1000 m}} \times \frac{\text{0.62 mile}}{\text{1 km}} \times \frac{\text{60 min}}{\text{1 hour}} = 9.9\frac{\text{mile}}{\text{hour}}.$$