Chapter 2
The Number System (Integers and Rational Numbers)

In this second chapter, students extend and formalize their understanding of the number system, including negative rational numbers. Students first develop and explain arithmetic operations with integers, using both tiles and the number line as models. Then, they move, in both models, to the rational numbers by observing that the unit represented by the tile, or the first hash mark on the number line, can be changed; for example it could be the qth part of the unit originally represented by the first tile, or the first hash mark on the line. As the laws of arithmetic remain the same, they extend directly to the arithmetic for the qth part of the original unit. It all holds together from the simple observation that q copies of the qth part of the unit returns us to the original unit. In this way, the arithmetic of rational numbers is a consequence of the properties of the four basic operations of addition, subtraction, multiplication and division.

By applying these properties, and by viewing negative numbers in terms of everyday contexts (i.e. money in an account or yards gained or lost on a football field), students explain and interpret the rules for adding, subtracting, multiplying, and dividing with negative numbers. Students re-examine equivalent forms of expressing rational numbers (fractions of integers, complex fractions, and decimals) and interpret decimal expansions in terms of successive estimates of the number representing a point on the number line. They also increase their proficiency with mental arithmetic by articulating strategies based on properties of operations.

In Grade 6 students come to understand that positive and negative numbers are used together to describe quantities having "'opposite" directions or values, learning how to place and compare integers on the number line. They use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation. Students recognize signs of numbers as indicating locations on opposite sides of 0 on the number line and recognize that the opposite of the opposite of a number is the number itself.

The development of rational numbers in Grade 7 is the beginning of a development of the real number system that continues through Grade 8. In high school students will extend their understanding of number into the complex number system. Note that in Grade 7 students associate points on the number line to rational numbers, while in Grade 8, the process reverses: they start with lengths and try to identify the number that corresponds to the length. For example, students find that the length of the hypotenuse of a right triangle is $\sqrt{5}$, they then try to identify the number that corresponds to that length.

The main work of this chapter is in understanding how the arithmetic operations extend from fractions to all rational numbers. At first addition is defined for the integers by adjunction of lengths, and subtraction as the addition of the opposite; in symbols: $a - b = a + (-b)$. This idea is a prelude to vector concepts to be developed in high school mathematics. Students' understanding of multiplication and division is extended to integers so that their properties continue to hold, using the understanding that the a number is the opposite of its opposite(e.g., $-(-1) = 1$).

Section 2.1: Add and Subtract Integers; Number Line and Chip/Tile Models

Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram. 7NS.1

Describe situations in which opposite quantities combine to make 0. For example, a hydrogen atom has 0 charge because its two constituents are oppositely charged. 7.NS.1a

Understand $p + q$ as the number located a distance $|q|$ from p, in the positive or negative direction depending on whether q is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts.7.NS.1b

Understand subtraction of rational numbers as adding the additive inverse, $p - q = p + (-q)$. Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this principle in real-world contexts.7NS.1c

We start the study of the operations of addition and multiplication for integers with a review of 6th grade, where the integers are placed on the number line. The reason for this is that, once students have understood the arithmetic operations with integers, in particular, the rules of signs, then the extension to all rational numbers is natural. This is because the rational numbers are placed on the line just by a change of unit from the unit interval to the $1/q$th interval for every positive integer q (see section 3). The only complication is the addition of rational numbers; this is discussed at the end of section 3.

The goal in this section is to build intuition and comfort with integer addition and subtraction so that by the end of the section students can reason through addition and subtraction of integers without a model, and then will be able to extend those operations to all natural numbers.

Students start by working with "opposites" (additive inverses) to notice that pairs of positives and negatives result in "zero pairs." They then move to adding integers. Students know from previous grades that the fundamental idea of addition is "joining." After reviewing prior understanding of subtraction, we turn the number line model of subtraction (as adding the opposite). Students should notice that joining positive and negative numbers often results in zero pairs and that which is "left over" is the final sum. Students will develop this idea first with a chip or tile model and the real line.

Rules for operating with integers come from the permanence of the rules of arithmetic, by which we mean that extensions of the concept of number require that the rules of arithmetic continue to hold. Students should understand that arithmetic with negative numbers is consistent with arithmetic with positive numbers.

When looking at a number line, two numbers are opposites when they are the same distance away from zero, but in opposite directions. Those numbers to the right of 0 are the positive numbers, and those on the left are the negative numbers. For example, "3" represents the point that is 3 units to the right of 0, and "−3" is its opposite, three units to the left. Two integers are opposites if they are each the same distance away from zero, but on opposite sides of the number line. One will have a positive sign, the other a negative sign. In the number line, +3 and −3 are labeled as opposites (it is customary to not write the plus sign in front of a positive number, so +3 will be denoted by 3.
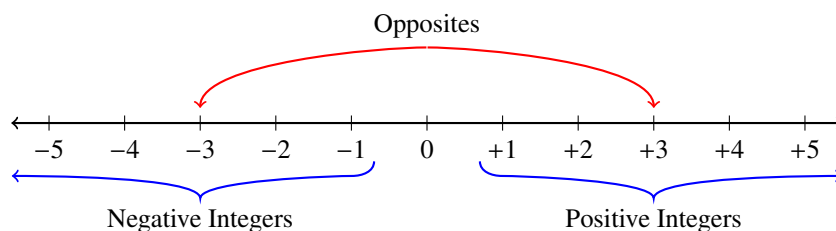
The full number line appears naturally in measurements, such as the thermometer, elevation, timelines, and banking. In each of these applications, we are making measurements "less than zero" of which the number line illustrates each of these concepts. Three observations that are important to notice:

> The integers are ordered: we say *a is less than b*, and write $a < b$, whenever b is to the right of a on the number line.
>
> The absolute value $|a|$ of an integer a is the distance from the point on the line to 0. A number and its opposite have the same absolute value.
>
> Every natural number has an opposite, or additive inverse. The negative integers are opposites of the positive integers.

For example, the opposite of 5 is -5, and the absolute value of both numbers is 5. Since the positive integers are the opposites of the negative integers, we conclude that $-(-5) = 5$, and in general $-(-a) = a$ for any integer a. The opposite of 0 is 0.
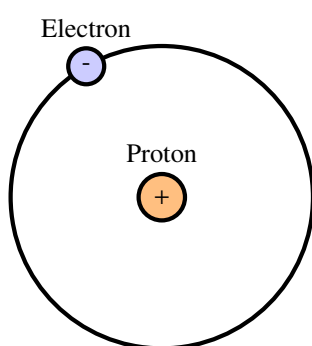
Hydrogen Atom. Source:
www.jwst.nasa.gov/firstlight.html

The idea of "opposite" occurs in the real world in many ways. The opposite of "income" is "debt." If I find $5, I can claim that I have +5 dollars; if I lose $5, that amounts to having −5 dollars. In chemistry a hydrogen atom, like other atoms, has a nucleus. The nucleus of a hydrogen atom is made of just one proton (a positive charge). Around the nucleus, there is just one electron (a negative charge), which goes around and around the nucleus. As a result, the "positive" proton and the "negative" electron balance the electrical charge, so that the hydrogen atom is electrically neutral (basically zero).

As with any new topic, it is important to start with familiar contexts so that students can use prior knowledge to build meaning. With integers, students often get confused about which symbol is the operation or in which direction they are moving when they compute, so having a context is particularly important. As students learn to compare and compute, they can use the contexts to ground their thinking and justify their answers.

In this course we will understand integers as points on the number line, and will also model the operations on integers with the chip/tile model. We emphasize the number line model because it is key to understanding the extension of arithmetic, and it provides the bridge between algebra and geometry that has been central to the development of mathematics for a millennium. The number line model amounts to working with a definition of addition and subtraction for integers and subsequently, rational numbers; and the chip/tile model (quantity) amounts to working with a theorem about addition and subtraction ($a - a = a + (-a) = 0$, represented by "zero pairs"), as a basis for understanding how the properties of operations should extend to integers.

A number line model has several advantages. In Grade 6 students learned to locate and add whole numbers, fractions and decimals on the number line. Additionally, the number line is an important connection to the coordinate axis, which involves two perpendicular numbers lines, explored in Grade 8.

Previously addition was represented by linking the line segments together. Moving from whole numbers to integers, we recognize that line segments have a beginning and an end; that is they are *directed*. So, now we have to be more clear: which end of one interval is linked to which end of the other? This clarification leads to an understanding of the relation of subtraction to addition and to "taking the opposite."

In the number line model arrows are used to show distance and direction. The arrows help students think of integer quantities as directed distances. An arrow pointing right is positive, and a negative arrow points left. Each arrow

is a quantity with both length (magnitude) and direction (sign). To put the integers on the line, we first select a point to the right of 0 and designate it as 1. The point at the same distance from 0, but to the left, is −1, the opposite of 1. Now, for a positive integer p, move to the right a distance of p units: the end point represents p. For a negative integer, do the same thing, but on the left side of 0. Now, we add integers on the line by adjunction of the corresponding directed line segments. To add the integers p and q: begin at zero and draw the line segment (arrow) to p. Starting at the endpoint p, draw the line segment representing q. Where it ends is the sum $p + q$. If p and q are of the same sign, the point $p + q$ is farther from the origin than q. If the second line segment is going in the opposite direction to the first, it can backtrack over the first, effectively canceling part or all of it out. The following figure depicts the sum $p + q$ where p is negative and q is positive.
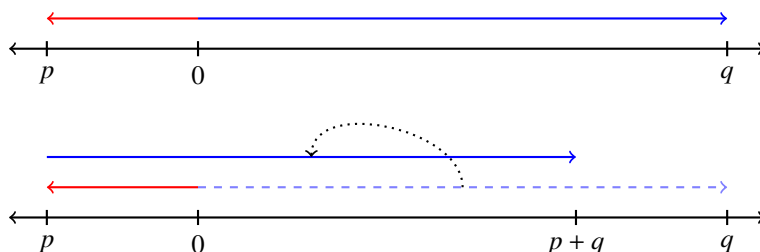

EXAMPLE 1.

Demonstrate each of the following on a number line.

a. $2 + 3$

**b.** $-4 + 3$

c. $12 + (-9)$

d. $-2 + (-3)$

SOLUTION. To avoid confusion, students should start by distinguishing between sign and an instruction for an operation. So, for example, in part c., the symbol "+" is an operation, whereas the "-" is a sign. This will continue to be an issue, so it is vital to start addressing this now. In the following we have used colors and arrowheads for stress, hoping that students will incorporate that in their understanding of vocabulary.

a. $2 + 3$: this addition can be thought of as starting at 0 and counting 2 units to the right (in the positive direction on the number line) and then "counting on" 3 more units to the right.
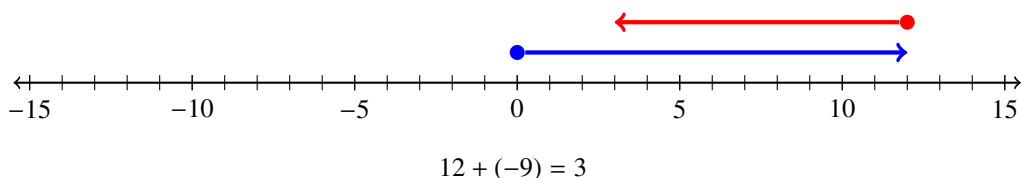

$$2 + 3 = 5$$
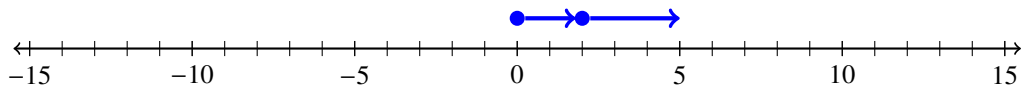
b. $-4 + 3$: this addition can be thought of as starting at 0 and counting 4 units to the left (in the negative direction on the number line) and then "counting on" 3 more units to the right.
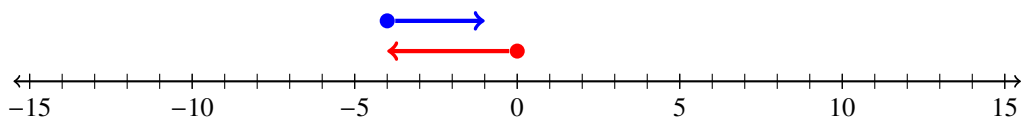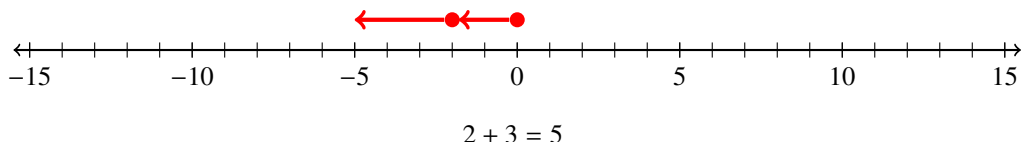

$$-4 + 3 = -1$$

c. $12 + (−9)$: start at 0 and count 12 units to the right, then count on 9 more units to the left.

$$12 + (−9) = 3$$

d. $−2 + (−3)$: Here we count 2 units to the left, and then 3 more units to the left.

$$2 + 3 = 5$$

Finally, note that if we add the opposite of a number to a number, we return to 0:

$$n + (−n) = 0 = (−n) + n \ .$$

for any number n . This equation states that n and $−n$ are *additive inverses,* that is, $−n$ and n add up to 0. Finally, note that parentheses can be used to avoid strange, or ambiguous expressions like $− − 3 = 3$ and $3 + −3 = 0$. Instead, write $−(−3) = 3$ and $3 + (−3) = 0$. Even 0 has an opposite, namely, itself, since $0 + 0 = 0$; that is, $−0 = 0$.

The second model mentioned utilizes colored chips/tile to represent integers, with one color representing positive integers and another color representing negative integers; subject to the rule that chips of different colors cancel each other out. Historically, Chinese traders used a system of colored rods to keep records of their transactions. A red colored rod denoted credits and a black colored rod denoted debits (opposite of how we view debits/credits today). Assign a red chip to represent $−1$ (debit or a negative charge) and a yellow chip to represent 1 (credit or a positive charge):

$$\blacksquare = \text{-1}, \quad \blacksquare = 1$$

A pair of red and yellow chips is called a *zero pair*, because it represents the integer 0. We use the informal term "zero pair" to refer to two things that are "opposite" or "undo each other". The sum of the two elements of a zero pair is zero. We see zero pairs in many contexts. For example, an atom with 5 protons and 5 electrons has a neutral charge, or a net charge of zero.

EXAMPLE 2.

I took 4 steps forward and 3 steps back. What is the result in words, and how many zero pairs are there?

SOLUTION. A visual model is shown and depicts that I am one step ahead from where I stared and that there are 3 zero pairs.


We emphasize that ▯ ▮ = 0, expressing $(-a) + a = 0$, the additive inverse property of integer addition. A red tile coupled with a yellow tile is a zero pair, that is, they cancel each other.

EXAMPLE 3.

Demonstrate the addition $-3 + 7$ with the chip/tile model.

SOLUTION. We begin with 7 yellow chips and 3 red chips in our first box. In the second box we recognize that there are three sets of zero pairs circled with each zero pair representing 0; resulting in (in the last box) four yellow chips or 4.
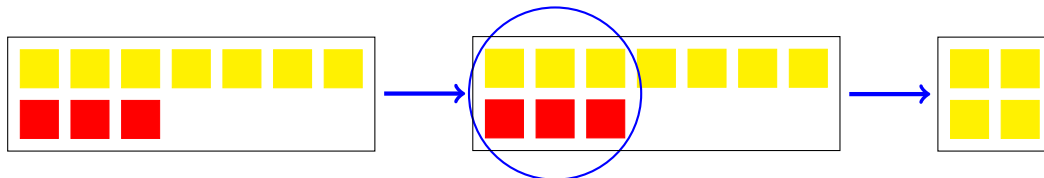
It is important to understand that it is always possible to add to or remove from a collection of any number of pairs consisting of one positive and one negative chip/tile without changing the value (i.e. it is like adding equal quantities of debits and credits). We also recognize that,

$$-3 + 7 = -3 + (3 + 4) = (-3 + 3) + 4 = 0 + 4 = 4 .$$

Although not explicitly stated, we are moving students towards understanding, either with the chip/tile model or the number line model the properties of arithmetic. Implicit in the use of chips is that the commutative and associate properties extend to integers, since combining chips can be done in any order.

Integer chips are useful in understanding operations with rational numbers, up to a point. By changing the unit concept from the interval $[0, 1]$ to the $1/q$th part of that interval (for any integer q) we can illustrate, with chips, some of the arithmetic facts for rational numbers. However, if we work with rational numbers with different denominators, this model becomes unwieldy. For this reason, in section 3, where we turn to arithmetic with rational numbers, we will work exclusively with the line model.

Next, students move to subtraction, first by reviewing, from previous grades, that there are two ways to think concretely about subtraction. a) "Take-away:" Anna Maria has 5 gummy bears. José eats 3. How many gummy bears does she now have? $5 - 3$, thus 2 gummy bears. b) "Comparison:" Anna Maria has 5 gummy bears, José has 3 gummy bears, how many more gummy bears does Anna Maria have than José? Again, the operation is $5 - 3$ resulting in 2 gummy bears. We build on the concept of comparison as a concrete way to think about subtraction with integers on the real line. We start by locating integers on the line and note that when comparing two integers, there is a directional or signed distance between them e.g. when comparing 5 and 3 we can think "5 is two units to the right of 3,"($5 - 3$ is 2), or "3 is two units to the left of 5," ($3 - 5$ is -2). In 2.1e students will examine subtraction exercises and notice that $a - b$ can be written as $a + (-b)$ and that $a - (-b)$ can be written as $a + b$:

$$a - b = a + (-b) \qquad a - (-b) = a + b .$$

As an example: if we subtract 11 from 7, we move 11 units to the left from 7, to -4. That is the same as adding -11 to 7.

EXAMPLE 4.

Model $7 - 2$ and $2 - 7$ on the number line.

SOLUTION. Think in terms of addition: $7 - 2 = 7 + (-2)$, and $2 - 7 = 2 + (-7)$.

a. Locate 7 and −2 on the number line. Take the directed line segment corresponding to −2 and move it so that its beginning point is at the point 7. Then the endpoint lands at 5.

b. Locate 2 and −7 on the number line. Take the directed line segment corresponding to −7 and move it so that its beginning point is at the point 2. Then the endpoint lands at −5.

Describing subtraction in terms of the number line model brings home the realization that integers involve two concepts: magnitude and direction, sometimes designated as "directional distance." We now see that there is a distinction between the distance from *a* to *b* and how you get from *a* to *b*, in other words, the directional distance.

Adding and subtracting integers can cause students a great deal of trouble particularly when they first confront exercises that have both addition and subtraction with integers. For example, in expressions like $-5 - 7$ students are often unsure if they should treat the "7" as negative or positive integer and if they should add or subtract. Put another way, students are not sure that $-5-7 = -5+(-7)$, and worry that it could mean $-(5-7)$, or even $-5-(-7)$. For this reason, it is extremely helpful that, from the beginning of this chapter, students articulate what they think the expression means.

We close section 2.1 with a few words of caution. Subtraction in the set of integers is neither commutative nor associative: $5 - 3 \neq 3 - 5$ because $2 \neq -32$; $(5 - 2) - 1 \neq 5 - (2 - 1)$ because $2 \neq 4$.

The expression $5 - 2 - 1$ is ambiguous because the order of operations is not specified. If we operate from the right we read this as $5 - (2 - 1) = 5 - 1 = 4$, and if we operate from the left, it is $(5 - 2) - 1 = 3 - 1 = 2$. But sometimes you will see an expression like 5-2-1, in which case it should be read as $(5 - 2) - 1$. That is: when the expression is ambiguous, the default is to operate from left to right. In any case, it is best to get in the habit of using parentheses to eliminate ambiguities.

Section 2.2: Multiply and Divide Integers; Represented with Number Line Model

Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers. 7.NS.2

Here we restrict attention to integers; in the next section we shall move to rational numbers, and deal with the full standard in that context. . The goal here for students in this section is twofold: 1) fluency with multiplication and division of integers and 2) understanding how multiplication and division with integers is an extension of the rules of arithmetic as learned in previous grades.

Multiplication of integers is an extension of multiplication of whole numbers, fractions and decimals.

Let us consider a typical elementary multiplication problem.

EXAMPLE 5.

Sally babysat for her mother for 3 hours. Her mother paid her $5 each hour. How much money did Sally earn after 3 hours?
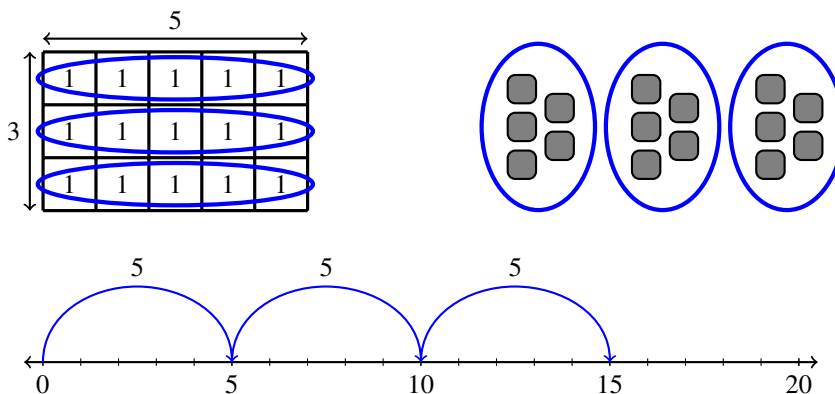
SOLUTION. We note that Sally earned $5 + $5 + $5 = $15, illustrated as

which is the sum of 3 fives, written as $3 \cdot 5$ and is read as "3 times 5," and means the total number of objects in 3 groups if there are $5 in each group.

Additional representations include:

Source: http://www.newmoney.gov/currency/5.htm

The basic definition of multiplication of whole numbers is, "if a and b are non-negative numbers, then $a \cdot b$ is read as "a times b," and means the total number of objects in a groups if there are b objects in each group." Note: if we rotate the rectangle in the first representation, we will have 5 groups of 3 objects in each group, representing 5×3. This is a way of seeing that $a \cdot b = b \cdot a$ for any positive a and b.

In elementary grade the dot symbol for multiplication is replaced by a cross, $\times$ (not to be mistaken for the letter x or x), or by an asterisk * (used in computer programming). Sometimes no symbol is used (as in ab), or parentheses are placed to distinguish the factors such as $(a)(b)$. In this text we will typically use the dot or parenthesis to express multiplication of expressions and the cross for numbers, to be sure that 3×5 is distinguished from 35.

We note that a and b are called factors, and that the first factor a stands for the numbers of groups, and the second factor b stands for the numbers of objects in each group (although, as we have pointed out, those roles are interchangeable).

How can you tell if the operation needed to solve a problem is multiplication? To answer this question we consider the definition of multiplication once again. Multiplication applies to situations that involve equal groups. In the illustration for our previous problem, there were 3 distinct groups of \$5 in each group. Thus, according to the definition of multiplication, there are 3 ? 5 dollar bills in all. Simply put, whenever a collection of objects is arranged into a groups, and there are b objects in each group, then we know that according to the definition of multiplication, there are $a \cdot b$ total objects. Up to this point we have examined multiplication only for whole numbers, so, what is the meaning of multiplication of negative numbers?

Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations, particularly the distributive property, leading to products such as (-1)(-1) = 1 and the rules for multiplying signed numbers. Interpret products of rational numbers by describing real-world concepts. 7.NA.2a

We will discuss this standard here in the context of integers, and then in the next section extend that logic to all rational numbers.

EXAMPLE 6.

What does $3 \times (-5)$ mean?

SOLUTION. Just as 3×5 can be understood as $(5) + (5) + (5) = 15$, so $3 \times (-5)$ can be understood as $(-5) + (-5) + (-5) = -15$.

On the number line, we think of 3×5 as 3 jumps to the right (or up) on the number line, starting at 0. Similarly, $3 \times (-5)$ is represented by 3 jumps of distance 3 *to the left* starting at 0, as in the following figure:
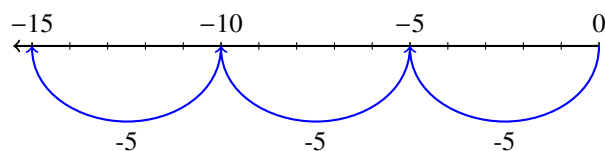

Using chip/tiles, $3 \times (-5)$ means 3 groups of 5 yellow tiles. And finally in context: if I lose \$5 on three consecutive days, then I've lost \$15.
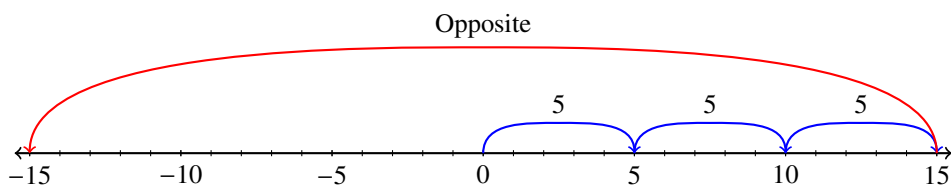
EXAMPLE 7.

What does $(-3) \times 5$ mean?

SOLUTION. There are two ways a student may attack this using rules of arithmetic: one way is to recognize that $(-3) \times 5$ is the same as $5 \times (-3)$ (e.g. the commutative property) and then apply the logic above: $-3 + -3 + -3 + -3 + -3 = -15$. A model would be illustrated as:
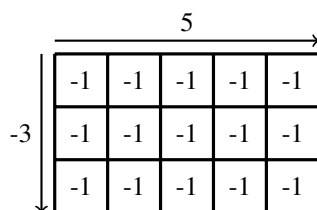
Another is to turn to the understanding of integers from 6th grade. There the idea of -3 was developed as the "opposite" of 3. Now , if the associate law of arithmetic extends, (the opposite of 3)×5 is the same as (the opposite of 3×5), or -15.

On the number line, we record $(-3) \times 5$ as the opposite of 3×5:

EXAMPLE 8.

Let us note that this argument gives us the equation $(-1) \times (-1) = 1$. For, multiplication by -1 is reflection to the opposite side of 0, and therefore lands on the opposite point. So, this equation is simply saying that the opposite of -1 is 1. It is instructive to go through the reasoning for this multiplication table:

$\times$	**1**	**-1**
1	1	-1
-1	-1	1

In other words, the product of two numbers with the same sign is positive, and if the numbers have opposite signs, it s negative. Notice that this is the same rule as that for the sum of even and odd numbers: the sum of two numbers of the same parity (both even or both odd) is even, if the parity is different, the sum is odd.

Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. 7.NS.A.2b

Just as the relationship between addition and subtraction helps students understand subtraction of rational numbers, so the relationship between multiplication and division helps them understand division. To make this more precise: for addition, the "opposite" of a number a is $-a$, the solution of the problem $a + x = 0$. On the number line, a represents going out from 0 to the point a, and adding $-a$ brings us back to the beginning. For this reason $-a$ is called the "additive inverse" of a. So, for multiplication, the "opposite" of a is the solution of the equation $ax = 1$, denoted $1/a$. On the number line, multiplying 1 by a means adding 1 to itself b times. To invert this process (that is, to get back to 1), we have to partition the result into a pieces of the same length; that is the partitive definition of division. So, dividing by a undoes multiplying by a, so $1/a$ is the "multiplicative inverse" of a; that is $a \times (1/a) = 1$, or $1/a$ is the solution to the equation $ax = 1$. It is this logic that extends from fractions to all rational numbers, as we shall see in the next section.

One important point: since $0 \times x = 0$ for every number x, there is no solution to the equation $0 \times x = 1$; and it is for this reason that *we cannot divide by zero*: it just plain does not make any sense.

In short, in terms of the number line, the properties of the operations of multiplication and division carry over from fractions to all numbers, positive and negative. Let us illustrate.

EXAMPLE 9.

      a. Calculate $(-15) \div 5$.

      b. Calculate $8 \div (-4)$.

      c. Calculate $(-12) \div (-3)$.

SOLUTION.

      a. Since division by 5 is the inverse of multiplication by 5, the equation tells us that the solution is some number x, which when multiplied by 5 gives us -15. But we know that $(-3) \times 5 = -15$, so the answer has to be -3.

      b. If $a = 8/(-4)$, then a is the solution of the problem $-4x = 8$. So, a has magnitude 2, but is it negative or positive? Let's check: $(-4)(2) = -8$ and $(-4)(-2) = 8$. So $a = -2$.

      c. $(-12) \div (-3)$ is that number which, when multiplied by -3 gives us -12. So, it has to be negative (for the product of two negatives is positive, so it can't be positive) and it has to be of magnitude 4. Therefor the answer is -4.

Section 2.3: Add, Subtract, Multiply, Divide Positive and Negative Rational Numbers (all forms).

Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram. 7.NS.1

Understand $p + q$ as the number located a distance |q| from p, in the positive or negative direction depending on whether q is positive or negative. Show that a number and its opposite have a sum of 0 (are additive inverses). Interpret sums of rational numbers by describing real-world contexts. 7.NS.1b

Understand subtraction of rational numbers as adding the additive inverse, p − q = p + (−q). Show that the distance between two rational numbers on the number line is the absolute value of their difference, and apply this

principle in real-world contexts. 7NS.1c

Apply properties of operations as strategies to add and subtract rational numbers. 7NS.1d

Apply and extend previous understandings of multiplication and division and of fractions to multiply and divide rational numbers.7NS.2

Understand that multiplication is extended from fractions to rational numbers by requiring that operations continue to satisfy the properties of operations. 7NS.2a

Understand that integers can be divided, provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number. If p and q are integers, then $-(p/q) = (-p)/q = p/(_q)$. Interpret quotients of rational numbers by describing real world contexts. 7NS.2b

Apply properties of operations as strategies to multiply and divide rational numbers. 7.NS.2c

Let's move on to rational numbers: placing them as points on the number line, and then extending the understanding of arithmetic operations to all natural numbers.

In the sixth grades students learned how to place the rational numbers on the number line, and we shall depend upon that knowledge in this section, although we will start anew: by defining and working with addition and subtraction of points on the line with a specific point identified as 0, by considering the point, and the line segment from 0 to that point as one and the same thing.

The points on one side of 0 are considered "negative" and those on the other side as "positive." Ordinarily the line is drawn as a horizontal line, and the positive points are those on the right of 0. If the line is drawn vertically, then the custom is to designate the segment above 0 as the positive segment. In some contexts the number line may be neither horizontal or vertical; it is important to realize that its position in the plane (or, for that matter, in space) is not relevant; all that matters is that a base point has been designated as 0, and one side has been designated as "positive." Once that has been done, the concept of "opposite" (the opposite of a is the symmetric point on the other side of 0). We use the notation $-a$ to represent the opposite of a. Without actually using the term *vector*. it helps to use this idea to understand *opposite*, and the arithmetic operations of addition and subtraction. For example, the act of "taking the opposite of a" is performed by reflecting in the origin, as illustrated in Figure 1.
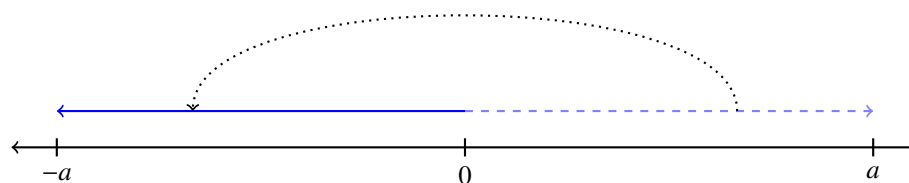

Figure 1. Taking the opposite: $a \to -a$

Addition, denoted $a + b$ for two points a and b on the line, is defined (vectorially) by adjunction: the sum of a and b is the endpoint of the interval formed by adjoining the interval to b to the interval to a (see figure 2).
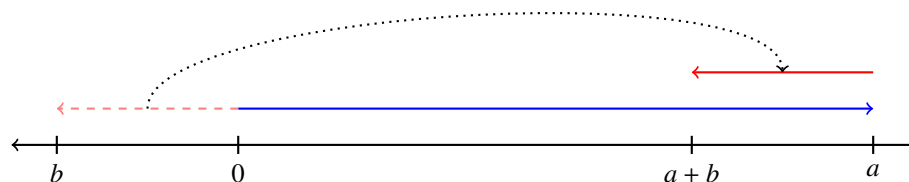

Figure 2. Addition: $(a, b) \to a + b$

7MF2-11

This brings the integers into the picture in this way: for any integer n, and any point a on the line, na is the endpoint obtained by appending the interval $[0, a]$ to itself n times. For $n > 0$, this is clear, and for $n < 0$ we mean the adjunction on the opposite side of $[0, a]$. In other words, for $a > 0$ and $n > 0$, na is the endpoint of the interval starting from 0 and moving to the right a distance of n lengths of size a, and $(-n)a$ is the endpoint of the interval starting from 0 and moving to the left a distance of n lengths of size a. This can be summarized by the equation (which works for either n or a either positive or negative:

$$(-n) \times a = n \times (-a) = -(n \times a)$$

Subtraction, denoted $a - b$, for two points a and b on the line, is defined as $a - b = a + (-b)$. Often, students will be confused by the symbol "$-$", since it represents both "subtraction" and "the opposite ." A way to deal with this confusion is to understand that it makes sense if we think of the opposite of a as obtained by subtracting a from 0: $-a = 0 - a$. In fact, in doing addition and subtraction, it is customary to delete the addend 0 if it appears. For example, one might replace $5 - 5 + a$ by $0 - a$ and then just omit the 0, getting $-a$. The basic fact that resolves the confusion is this: if b is subtracted from a, that is the same as adding $-b$ to a; that is, $a - b = a + (-b)$. Furthermore, $-(-a) = a$, and $a - (-b) = a + b$.

Having defined addition and subtraction for points (as intervals) on the real line, we can pictorially demonstrate that the properties of addition extend naturally to the same operations on the line. Here are two examples; students can reinforce their understanding of those operations by drawing diagrams for other laws of arithmetic.
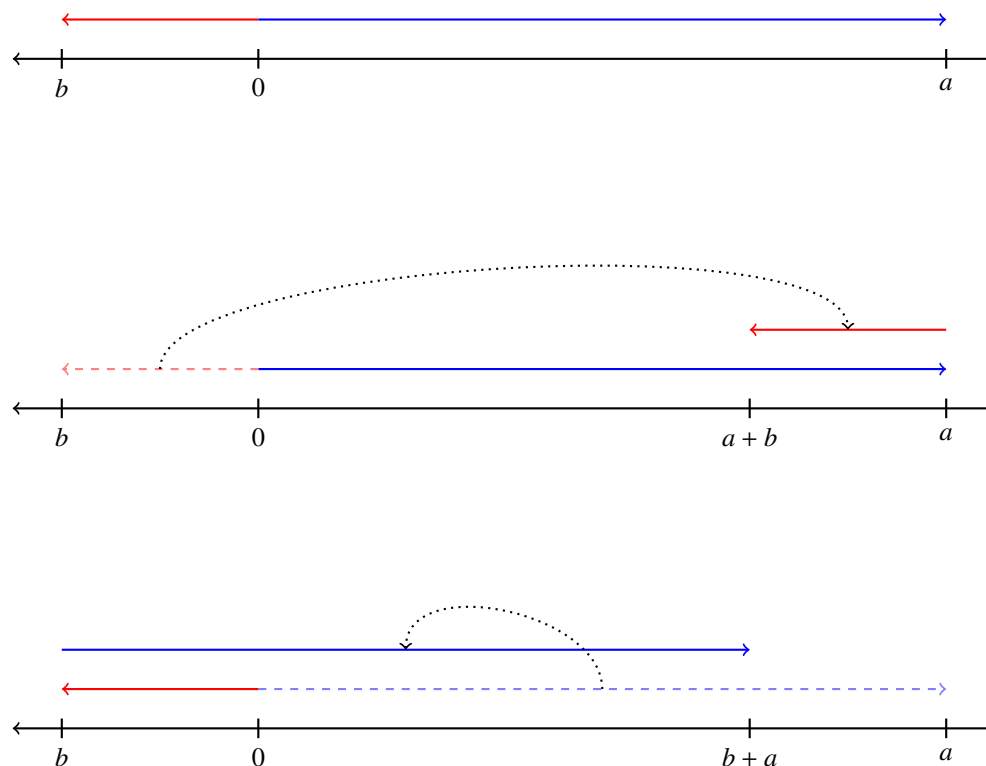
EXAMPLE 10.

a + b = b + a

Figure 3. $a + b = b + a$

EXAMPLE 11.

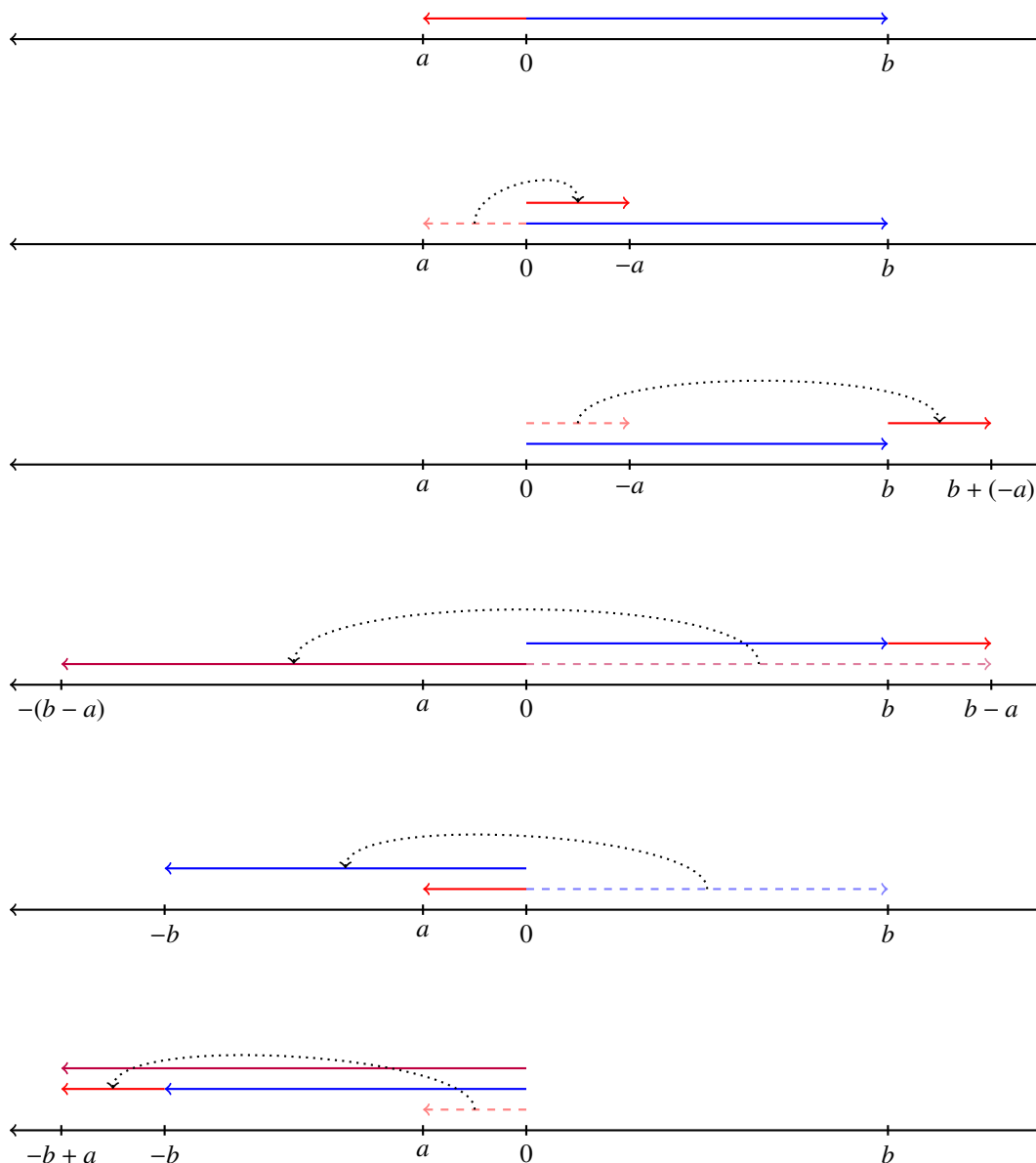$-(\mathbf{b} - \mathbf{a}) = -\mathbf{b} + \mathbf{a}$

Figure 4. $-(b - a) = -b + a$

Bringing this discussion together with that of the previous section, we can summarize in this way:

Take a line, and a point 0 on the line. Let's assume the line is horizontal. The point 0 divides the line into two segments; the "positive side" on the right of the point 0, and the "negative side" on the left. Define *opposite* and *addition* as above. There is a relation on this line: the expression $a < b$ means that the point a lies to the left of b. This is the same as saying $b - a$ is positive. For any two different points a, b, either $a < b$ or $b < a$. If $a < b$ and $b < c$, then $a < c$. For any a and b, $a \leq b$ the segment, or interval, $[a, b]$ is the set of points between a and b, ordered from a to b. In particular, $[a, a]$ is just the point a. Finally we define addition of points a and b by adjunction of the intervals $[0, a]$ and $[0, b]$: put the 0 endpoint of $[0, b]$ at the 0 endpoint of $[0, a]$: the combined interval is the interval $[0, a + b]$.

Now select a point on the right of 0, and denote it as 1. Then, for any positive integer n, n is represented by the sum of n copies of the interval between 0 and 1 (where, for n negative, we do the same on the other side of the line starting with the opposite -1 of 1).
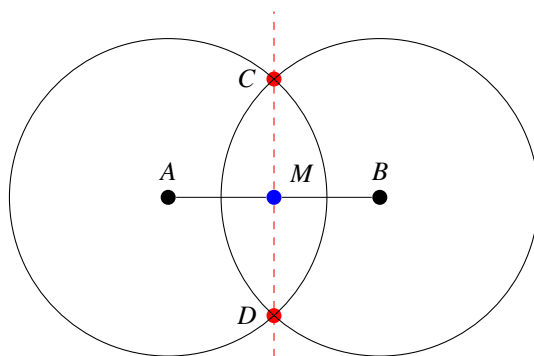

Figure 5

Finally, we put rational numbers on the line. Given p/q, where p is an integer and q is a positive integer, first partition the interval $[0, 1]$ into q equal parts, and denote the first interval as $[0, 1/q]$. If we adjoin q copies of this interval to each other, we end up at the point 1, showing that $q(1/q) = 1$. Now p/q is the endpoint of the interval obtained by adjoining p copies of $[0, 1/q]$ to each other (on the left side of 0 if $p < 0$. Incidentally, the instruction "partition the interval $[0, 1]$ into q equal parts" can actually be done by construction with straight edge and compass. For $q = 2$, students probably already know how to bisect a line segment (see figure 5 for a reminder); a different construction for any positive integer q will be learned in high school mathematics.

Multiplication and division of rational numbers, as points on the line, are defined geometrically just as they are on the integers, with the requirement that the rules of arithmetic persist. Let us explain:

First of all, if p is a positive integer and a is a point on the line, Then $p \times a$ is defined as the adjunction of p copies of the interval $[0, a]$ from 0 to a. We can extend the definition for $p = 0$: $0 \times a$ represents zero copies of a; in other words $0 \times a = 0$. Now, if p is a negatiive integer, $p \times a$ is defined as the adjunction of p copies of the opposite of a. Now, if q is a positive integer a/q is defined as the first piece of a partition of the interval $[0, a]$ into q pieces of the same length.

This suffices to define multiplication of any length by an integer, and division of any lengths by a positive integer. But we want to show that we can multiply any two rational numbers, and divide any rational number by a nonzero rational number. To get there takes a few steps.

We are used to the understanding (for a positive integer p, and an interval $[0, a]$, that $p \times a$ represents p copies of $[0, a]$ adjoined together sequential, and $(1/q) \times a$ represents one qth part of $[0.a]$. But that is how we defined a/q, so

$$\frac{1}{q} \times a = \frac{a}{q} \ .$$

Now, how do we multiply any point a on the line by a rational number p/q? If the laws of arithmetic are to hold, we must have

$$\frac{p}{q} \times a = p \times (\frac{1}{q} \times a) = p \times \frac{a}{q} \ .$$

In words, $(p/q) \times a$ is constructed in this way: First divide a into q equal parts and then add together p copies of that length. Now, we might just has well have written

$$\frac{p}{q} \times a = (\frac{1}{q})(p \times a) \; ;$$

in words:: , $(p/q) \times a$ is constructed in this way: First adjoin together p copies of a, and then divide the result into q equal parts. One of the strengths of arithmetic is that these two interpretations are correct, and result from the commutativity of multiplication.

When we defined division, $a \div b$, we said that the divisor b should be not zero. Why not?

We know that division is the inverse operation of multiplication. For example, the unique (one and only one) solution of 15/5 = 3, since 3 is the value for which the unknown quantity in (?)(5) = 15 is true. But the expression 15/0 requires a value to be found for (?) that solves the equation: (?)(0) = 15. But any number multiplied by 0 is 0 and so there is no number that solves the equation.

Extension

The above is just one of many explanations of the prohibition of division by 0. But students have seen many instances in which an arithmetic gap is filled by inventing new numbers. Maybe we can introduce a new number, call it ξ, that solves this equation, that is $\xi \times 0 = 15$. But if the laws of arithmetic are to continue to persist, then we'd have $15 = \xi \times 0 = 0$, which is a contradiction: 15 is not the same as 0. The persistent student my ask that we might succeed by taking away the rule $x \times 0 = 0$. . This would force us to have a specific ξ for every number, and this shortly leads us into new contradictions. Further exploration of this way is doomed to failure: there is no way to try to define division by 0 unless we want to sacrifice the rule that division is the inverse of multiplication, in which case, we are no longer talking about the arithmetic of the number line.

The point here is this: if we want to extend the number field, it should be based on a specific context; in our case: the number line. Let's look at the number line context for our ξs. Locate 15/a with $a > 0$ on the number line. As a gets smaller and smaller, 15/a gets larger and larger, and if we slide a to zero, 15/a slides off the number line on the right. So, 15/0, if it can make any sense at all, is off the number line. Until we find a new context where this makes sense, we have to abandon the hope to define division by zero.

End Extension.

But, now, what sense are we to make of division by fractions? What is the answer to the question

$$\frac{\frac{1}{5}}{\frac{1}{7}} = ?$$

Let the laws of arithmetic be our guide: the question mark s filled with a number that has this property: when multiplied by 1/7, we get 1/5. That is the same as saying: the question mark s filled with a number that has this property: when divided 7, that is, when partitioned into 7 equal parts, each part is a copy of 1/5. The answer then is 7 copies of 1/5, or 7/5:

$$\frac{\frac{1}{5}}{\frac{1}{7}} = \frac{7}{5} \; .$$

In conclusion, these constructions associate to every rational number (quotient of an integer by a positive integer) a point on the line.

Extension

Does this give us *every* point on the plane? This is a question that is posed and discussed in Grade 8, but the persistent 7th grader may start wondering about this.

In practice, a point is a blot on the line, and so has positive length. So, we can get within that length with the decimal approximation of the point (that is, getting within the nearest integer of the point, then the nearest tenth, then the nearest hundredth etc.). If all we care about is naked-eye accuracy, we might need two or three decimal points. For accuracy with a magnifying glass, we;ll need more; with a microscope, much more, and so forth. No matter how good our instrumentation, a point will appear as a blot. Plato, who invented the concept of the ideal, conceived of this process going on indefinitely: although mortals will always only see a blot, the Gods can actually see the point. Aristotle was more pragmatic: all there ever is is a blot, but it may get finer and finer.

Another way to express an answer is through decimal expansions. The 7th grader is introduced to never-ending decimals, so we start the discussion there Given any point, our construction of the decimal approximation to the point can get us as close as we need to the point (as 0.333333333 is close to 1/3, and if we append more 3's we continually get closer - but never there). To paraphrase Plato's conception (a paraphrase since decimal expansions were not known at Plato's time) - on the contrary: $0.\overline{3}$ *is* there. Well that's ok, since 1/3 is a rational number.

But what about

$$0.101001000100001000001\cdots ?$$

Here "..." means that the pattern continues indefinitely; the patter being that each 1 is followed by one more 0 than the preceding 1, going on indefinitely. Now a rational number (as we have seen in chapter 1) has a decimal expansion that is either terminating or repeating. So, if this represents a number, it cannot be rational. So maybe the answer is that all points on the line are represented by decimal expansions, and every decimal expansion does describe a point on the line. It is extremely useful to discuss this idea through the understanding of a decimal expansion as a recipe for getting as close as wee need to be to a point on the line.

The issue of making mathematical sense of Plato's ideal was not resolved until the 19th century. The focus was on the questions: what is number? what does it mean that a line consists of points? In the classical Greek days, line and point were undefined terms, and number was for counting. In the number line, number means a (signed) length: a is the signed length of the interval between a and 0 (positive is a is right of 0, negative if a is left of zero). Now, it was known by the Pythagoreans long before Plato that there exists a length (in the sense of *constructable by straightedge and compass*) that is not representable by a natural number. Let us begin with the problem as it was understood at the time.

The discovery of the Pythagoreans is this: the diagonal and side of a square are *incommensurable*. Euclid (about a century after Plato) defined this term in this way. Take two intervals $A = [0, a]$ and $B = [0, b]$. A and B are commensurable if there is another interval R such that both A and B are integral multiples of R; that is, there are integers m and n such that $A = mR$ and $B = nR$. In particular, if B is the unit interval, $[0, 1]$, then A corresponds to the rational number m/n. If no such interval R exists, we say that A and B are incommensurable. As mentioned earlier, the Pythagoreans found out that the side and diagonal of a square are incommensurable; Euclid, in his *Elements*, gave a geometric proof of this fact, and Aristotle gave an arithmetic proof (both these will be further discussed in 8th grade).

Euclid based his argument on a test for commensurability that he invented, called the *Euclidean algorithm.* It goes like this: again, let A and B represent two (positive) lengths; let us suppose that $A > B$. Now we mimic the decimal expansion of A using B as the unit measure: start adjoining copies of B inside the interval A until we cannot add another copy without going beyond A.If we end up right at the end of A, we are done, for we have shown that A is an integral multiple of B. If not, let B_1 be the interval remaining after this fill. Since $B_1 < B$, we can start filling B with copies of B_1. If we end precisely at the endpoint of B, we are done: both A and B are integral multiples of B_1. If not, let B_2 be the piece left over; since $B_2 < B_1$, we can try to fill B_1 with an integral multiple of B_2s, and so forth. If this process ends at any stage, the lengths A and B are commensurable; if not, they are incommensurable.

As a check on this procedure, we note that integer lengths are commensurable, since they are both integral multiples of the unit length. Thus, in this case, Euclid's test must end. And it does: when we put a copy of B inside A, since both A and B are integers, what remains is an integer also; in fact the integer $A - B$. So when we've put as many copies of B inside A as is possible, what remains B_1 is an integer less than B. Continuing this process, we get a decreasing sequence of integers, starting with B, so it must end. The remarkable thing is that the integer R at which this process ends is the greatest common divisor of A and B.

How could Euclid possibly show that the side S and the diagonal D are incommensurable; if he tried to keep repeating the process, he'd still be there in his little attic working away, and without a proof. Well: he was clever: he showed this: if we put the length S inside D, we have a piece P left over that is less than S. Now, he notices that P cannot fill S, and that the piece left over is smaller than P; but what is more, the geometry of the original square has been reproduced, only as a smaller version. So, since the geometry reproduces, the process can never end.

Let us give another example: that of the golden rectangle. A *golden rectangle* is a rectangle that is not a square, but has this property: if we remove the square of whose side is the length of the smaller side of the rectangle, the remaining rectangle is a smaller version of the original. Clearly, if there is a golden rectangle, its sides cannot be commensurable. For, when we take away the square based on the smaller side, what remains is a reproduction of the original, at a smaller scale. This tells us that this process will never end.

End Extension.

Now, let's return to the representation of rational number p/q on the number line. We have been able to geometrically describe the arithmetic operations, but still, the sum of two rational numbers requires some explanation. Since the expression a/b is taken to mean a copies of $1/b$, then it is easy to add (or subtract) rational numbers with the same denominator: a copies of $1/b$ plus c copies of $1/b$ gives us $a + c$ copies of $1/b$:

$$\frac{a}{b} + \frac{c}{b} = \frac{1}{b}a + \frac{1}{b}c = \frac{1}{b}(a + c) = \frac{a + c}{b} \ .$$

Now, to add $a/b + c/d$ with $b \neq d$, we proceed as follows. First, divide the unit interval into bd equal pieces. Note that $1/b$ consists of the first d copies of $1/bd$, and $1/d$ consists of the first b copies of $1/bd$. In other words, the fractions $1/b$ and d/bd represent the same length, as do $1/d$ and b/bd. Thus:

$$\frac{a}{b} + \frac{c}{d} = a\frac{1}{b} + c\frac{1}{d} = a\frac{d}{bd} + c\frac{b}{bd} = \frac{ad}{bd} + \frac{cb}{bd} = \frac{ad + bc}{bd} \ .$$

EXAMPLE 12.

$$-\frac{2}{5} + \frac{3}{7} = \frac{-14}{35} + \frac{15}{35} = \frac{-14 + 15}{35} = \frac{1}{35} \ .$$

EXAMPLE 13.

How much is 2/5 of \$55.35?

SOLUTION. There are several ways to do this. Here is one:

$$\frac{2}{5}(55 + \frac{35}{100}) = \frac{2}{5}55 + \frac{2}{5}\frac{35}{100} = 22 + \frac{14}{100} = 22.14 \ .$$

Let's take a second look at these computations. In the first fraction, divide 55 by 5 too get 11, and then

multiply by 2 to get 22 (dollars). For the second fraction, divide 35 by 5 to get 7 and multiply 2 to get 14 (cents). Another way is to observe that $2/5 = 0.4$, and then multiply:

$$0.4 \times 55.35 = 0.4 \times 50 + 0.4 \times 5 + 0.4 \times .35 = 20 + 2 + 0.14 = 22.14 \,.$$

Solve real-world and mathematical problems involving the four operations with rational numbers.(7.NS.3)

EXAMPLE 14.

Our group shared 3 pizzas for lunch. The pepperoni pizza was cut into 12 equally sized pieces, the tomato pizza into 8 pieces and the broccoli pizza into 6 pieces. I ate one piece of each. What fraction of a whole pizza did I eat?

SOLUTION. My consumption was $1/12 + 1/8 + 1/6$. Now, I don't have to multiply $12 \times 8 \times 6$ since each denominator is a multiple of 24 : $24 = 2 \times 12 = 3 \times 8 = 4 \times 6$. Thus

$$\frac{1}{12} + \frac{1}{8} + \frac{1}{6} = \frac{2}{24} + \frac{3}{24} + \frac{4}{24} = \frac{9}{24} = \frac{3}{8} \,.$$

EXAMPLE 15.

Genoeffa left her house and walked to the East down her street for a mile and a quarter. She then turned around and walked to the West for two and thee-eights miles. How far was she from her home?

SOLUTION. We can think of her street as the number line, with her house at the zero position. Genoeffa walked in the positive direction to the point $1\frac{1}{4}$, and then walked in the negative direction a distance of $2\frac{3}{8}$. Using the number line model, her new position is at the point

$$1\frac{1}{4} + (-2\frac{3}{8}) \,.$$

Mixed fractions are often convenient in everyday discourse, but are very inefficient in a computation. So we rewrite the expression as

$$\frac{5}{4} - \frac{19}{8} \,.$$

Put both fractions over a common denominator, in this case 8, and subtract for the answer:

$$\frac{10}{8} - \frac{19}{8} = -\frac{9}{8} \,,$$

so Genoeffa ended up a mile and an eighth west of her house.

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