Chapter 8
Measurement in 2-3 Dimensions, Cross-Sections of Solids

Geometric and spatial thinking connect mathematics with the physical world and play an important role in modeling phenomena whose origins are not necessarily "physical." An example of this is the use in 6th grade of *Nets* in the context of area and volume. Geometric thinking is also important because it supports the development of number and arithmetic concepts and skills, by providing students with a context for intuitive understanding. The sections in this chapter emphasize key ideas that assist students in developing a deeper understanding of numbers. In grades K-6 students learned to work with basic two-dimensional geometric shapes: triangles, squares, rectangles, and others. In addition, students learned specific parts and properties of shapes allowing them to identify and classify them into categories, and to determine how the categories of shapes are related. In this chapter students will be engaged in using what they have previously learned about drawing geometric figures (using rulers and protractor, coordinate grids and technology) to solve problems involving area and circumference of a circle, and real-world mathematical problems involving area and perimeter of two-dimensional objects composed of triangles and quadrilaterals. Furthermore, students will explore 3D geometric figures and circles and apply their mathematical knowledge of rational numbers, proportionality and algebra to solve problems involving surface areas and volumes, and to express meaningful formulas and recognize equivalent expressions.

More specifically, section 8.1 builds from understandings of geometry, measurement and data from grades 3-6. It utilizes the scope of the number system and is a review and extension of previously learned skills. For example, in sixth grade students learned how to find area by creating rectangular arrays. Using the shape composition and decomposition skills, students learned to develop area formulas for parallelograms and triangles. They also learned how to address three different cases for triangles: a height that is a side of a right angle, a height that lies over the base and a height that is outside the triangle. Composition and decomposition of regions continues to be important for solving a wide variety of area problems, including justifications of formulas and solving real world problems, as we will see in section 8.1. We will further see that composition and decomposition of shapes is important since it is used throughout geometry from Grade 6 to high school and beyond.

Previously, in Chapter 5, students learned how to find the circumference and area of circles, whereas the focus of section 8.1 will be to extend and apply the area and perimeter of circles, triangles, rectangles, parallelograms, and trapezoids to various real-world and mathematical problems. Our goals for section 8.1 will be: i) solving problems involving area and circumference of a circle, ii) solving real-world and mathematical problems involving area and perimeter of two-dimensional objects composed of triangles and quadrilaterals, yet most importantly contrasting and relating area and perimeter.

Our focus for section 8.2 will center on 3D figures. Students begin by examining plane sections of 3D figures. The point of work in the elementary grades with plane sections was to develop the ability to use drawings and physical models to identify parallel lines and planes in 3D shapes, as well as lines perpendicular to a plane, lines parallel to a plane, and to be able to construct the plane passing through three given points, and the plane perpendicular to a given line at a given point. For this reason, in the elementary grades, plane sections were actually *cross sections*: special plane sections parallel to a face of the object, or perpendicular to an axis of symmetry of the object. (We note that it has become customary to use these names interchangeably). In grade 8 we want to go more deeply in the detailed visualization of 3D objects, and for that reason, we consider all sorts of plane sections.

Furthermore, in the elementary grades, students study volume and surface area of special objects in a descriptive way. In 7th grade we want to go further, in order to understand the distinctions and relations between surface area and volume. As the volume of an object grows, does its surface area grow? This is the analog in 3D of the study of perimeter and area of figures in the plane. Here we introduce the ideas involved in computation of volumes, and then relate that to the determination of surface area using nets (as in 6th grade). Students will then differentiate between surface area and volume and use their understanding to solve various problems.

One of the tools introduced at this point is Cavalieri's principle: that the volume of a figure developed around a particular axis is determined by the area of the section of the object by planes perpendicular to the axis. This is not a grade 8 core topic, but it seems to fit naturally and easily in the discussion of sections, to provide an added intuition into area calculations.

Section 8.1: Measurement in Two Dimensions

Solve real-world and mathematical problems involving area, volume and surface area of two- and three- dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms. 7.G.6.

Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle. 7.G.4.

Throughout this chapter, as in Chapter 5, students and teachers use geometric terms and definitions with which they have becomes familiar: polygons, perimeter, area, volume and surface area of two-dimensional and three-dimensional objects, etc. Though these terms are not rigorously defined, it is important that they are used correctly and misconceptions are not allowed to develop. For this reason we start by reviewing the frame for using geometric terms. Something that we cannot stress too much is that to "know the formula" does not mean memorization of the formula. We are striving for an understanding of why the formula works and how the formula relates to measure (length, area and volume) and the figure. The surface area formulas are not the expectation with this standard; the expectation is that students will understand the process and procedures for determining those formulas.

A central construction of objects in three dimensions is that of drawing a planar figure out in the third dimension. This creates the parallel with two dimensions: just as area in 2D is the product of length and the distance this length is drawn out, volume in 3D is the product of area with the distance drawn.

A *polygon* is a planar figure consisting of a sequence of line segments with the property that the initial point of each line segment is the end point of the previous line segment, and the endpoint of the last segment is the initial point of the first segment. These endpoints are the *vertices* of the polygon. A *triangle* is a polygon with three sides, and a *quadrilateral* has four sides. Before moving on to more detailed vocabulary, it is necessary to point out an ambiguity, which in this text will be resolved by the context. By this definition, a triangle consists of the set of line segments that act as the boundary of a region in the plane. When we speak of the *area* of a triangle, we mean the area of the region bounded by the triangle. Similarly *circle* - the set of all points equidistant from one point, called its *center* - refers to the boundary; and when we speak of the area of the circle, we mean the area of the region bounded by the circle (to which we sometimes refer as the *disk* bounded by the circle).

Now, different kinds of triangles are defined by adjectives: acute, scalene, right, isosceles, etc. But for quadrilaterals, we have different nouns: square, rectangle, parallelogram, etc. This issue arises: when we say the word "rectangle" do we mean a rectangle that is not a square, or are squares included? Ordinarily, the designations are meant to be included: a square is a particular kind of rectangle. For clarity, let's define the various kinds of quadrilaterals, starting from the most inclusive:

- A *quadrilateral* is a polygon with four sides.
- A *trapezoid* is a quadrilateral with one pair of parallel sides .
- A *parallelogram* is a quadrilateral with both pairs of opposing sides parallel.
- A *rhombus* is a parallelogram with all sides of the same length.
- A *kite* is a quadrilateral with two pairs of adjacent sides of the same lengths.
- A *rectangle* is a parallelogram with at least one right angle.
- A *square* is a rectangle with all sides of the same length.

See Figure 1 for images of these different categories of polygons.

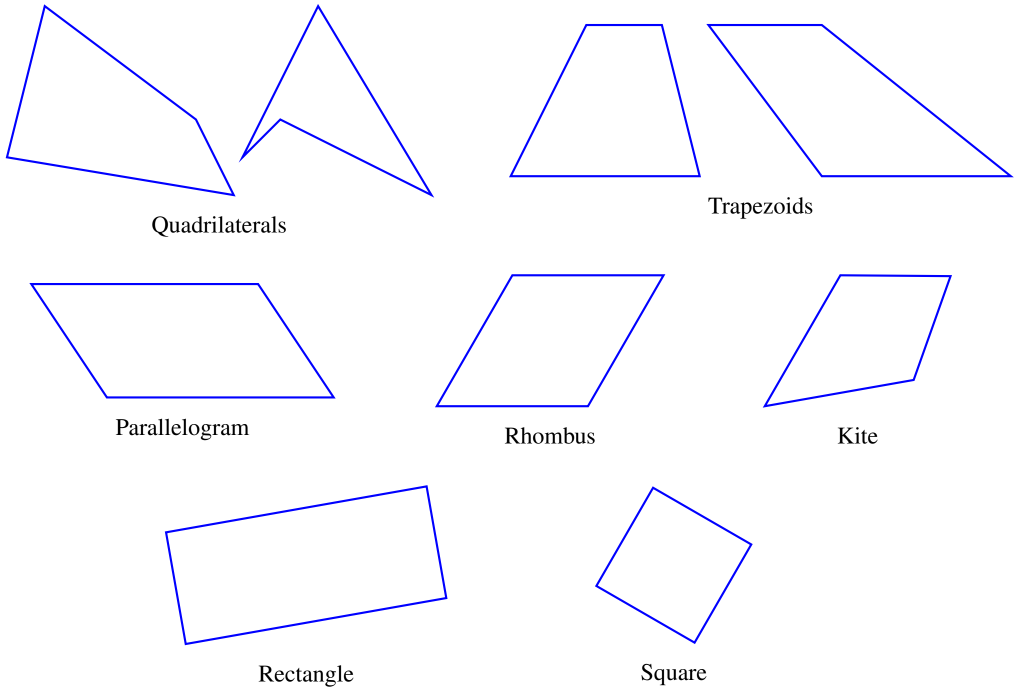
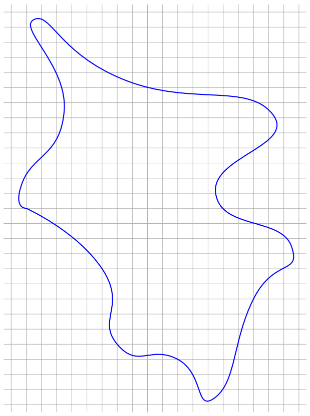


Figure 1

Area



On the plane, *area* refers to the measure of the region enclosed by a curve (see the accompanying figure). For this to make sense, we need to have specified a unit measure of area. This is usually accomplished by putting a coordinate grid on the plane where each coordinate square is assigned the area of 1 square unit. Then the number of coordinate squares that is contained inside the curve is an *approximation* of the area of the region, in square units. To get a better approximation to the area of the region, we can create a finer grid, so as to come closer to the boundary of the region, and make the same count, but now multiplying that count by the area of the new grid squares. For example, suppose that, in the attached figure, the measure between any two horizontal lines and two vertical lines is one cm. Then the measure of each square is 1 sq. cm. We count 165 squares in this figure, and conclude that an approximation to the area of the figure is 165 sq. cm.

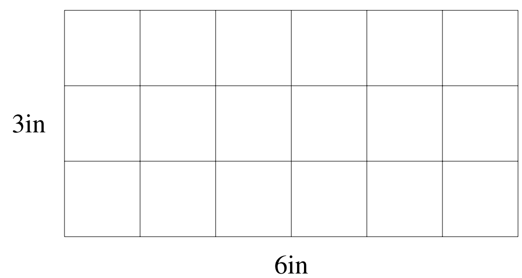
Now, let's refine the grid by putting in all the lines through the midpoints between any two lines in the grid. Now, each square of the original grid contains 4 squares of the new grid, so the size of each square in the new grid is

(1/4) sq. cm. Again we count the number of squares in the new grid contained inside the curve, which we find to be 698. Since each is of area 0.25 sq. cm, the new approximation of the area is 0.25) $\times$ 698 = 174.5 sq. cm. If we need a better approximation, we can make the grid more fine, count the squares inside the curve and multiply that number by the area of the refined squares.

Note that this procedure provides a concrete illustration of the concepts of scale and proportion: dividing the unit into four equivalent pieces requires us to divide the count of new squares inside the curve by four so as to keep the units consistent.

This procedure can be simplified for particular figures, using simple properties of area. Most important of which is this: if a region is subdivided into two pieces that do not overlap, then the area of the region is the sum of the areas of the two pieces. This simple rule allows us to find the area of regions enclosed in a polygon.

How do we figure out the area of a polygonal shape? What does it mean to say that the area of a region is 18 square inches? It means that the shape can be covered, without gaps or overlaps, with a total of eighteen 1-inch-by-1-inch squares, allowing for squares to be cut apart and pieces to be moved if necessary. If the figure is a 3×6 rectangle, we can cover it with 1×1 squares, and count the squares: there are 18 of them. In fact, students will recall that this is the geometric intuition that led to the concept of multiplication: 3×6 *is* the area of a rectangle with side lengths of 3 units and 6 units.



3 rows with 6 squares, each of area 1 in, in each row, so the area is 3 in×6 in.

How about a general polygonal figure? In general, the technique for calculating areas of general polygonal figures, or formulas for specific types of polygons, is based on these principles:

1. If you move a shape rigidly (without stretching or distorting it), then its area does not change.
2. If you combine (a finite number of) shapes without overlapping them, then the area of the resulting shape is the sum of the areas of the individual shapes.

Now, let's recollect area formulas to which students have already been exposed in the elementary grades. The point here is to demonstrate that they all come about, starting with the basic definition, and using the basic properties of area (**1** and **2** above).

Example 1. Rectangle.

If the lengths of the sides of a rectangle are a and b units, then its area is ab square units.

Example 2. Triangle.

To find the area of a triangle: designate one its sides as the *base*, and denote its length by b. The distance from the base to the opposing vertex is called the *height* of the triangle, denoted h. Then the area of the triangle is $\frac{1}{2}bh$.

If the triangle is a right triangle, we can designate one leg to be the base (of length b) and the other to be the height (of length a). If we rotate the triangle around its hypotenuse, we obtain a rectangle consisting of two copies of the given triangle, whose area is bh (see Figure 3), so the area of the triangle is half that.

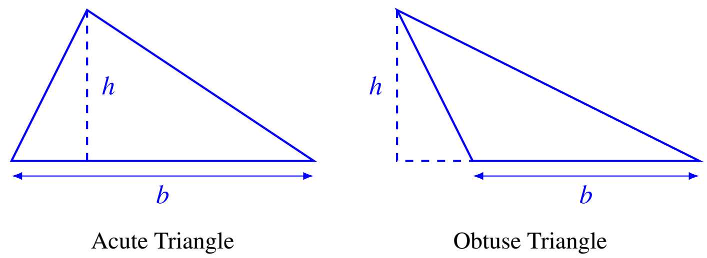


Figure 2

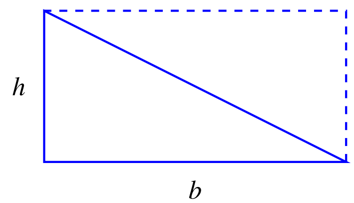


Figure 3

Now, referring back to Figure 2, when we drop the perpendicular we divide the triangle into two right triangles, the sum (or difference, depending upon whether the original triangle is acute or obtuse) of whose bases is b, so the general fact holds.

EXAMPLE 3. PARALLELOGRAM.

Choose one side of the parallelogram, call it the *base* and its length b, and let h be the distance between the base and its opposite side. (See Figure 4). Then, the area of the parallelogram is bh.

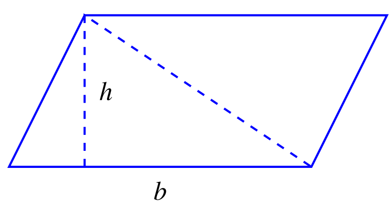


Figure 4

If we now draw a diagonal of the parallelogram, it divides the parallelogram into two triangles, each of which has area $\frac{1}{2}bh$, so the result follows.

EXAMPLE 4. TRAPEZOID.

Let the lengths of the parallel sides be b_1 and b_2, and the distance between them h. Then the area of the trapezoid is $\frac{1}{2}(b_1 + b_2)h$.

Rotate the trapezoid around the vertex V for 180°, as shown by the red circular arrow. Now translate the new trapezoid as indicated by the straight red arrow, so that the lower bases of the two trapezoids are on the same line. The result is a parallelogram of height h and side length $b_1 + b_2$. Thus its area is $(b_1 + b_2)h)$. Finally, the original trapezoid is precisely half of this parallelogram, and so its area is $\frac{1}{2}(b_1 + b_2)h$.

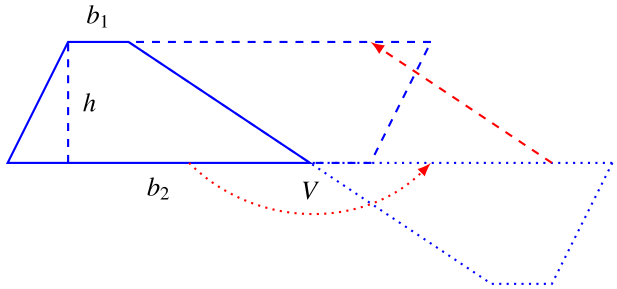


Figure 5

The point of the preceding is not just to bring together known facts, but now justifying them by general principles, but also to start students thinking in terms of transformations. So, for example, in case of the trapezoid, there are several clever ways of relating the new computation to known ones. We provide one that is based on moving figures around, rather than constructing them, precisely for this purpose.

Example 5.

Find the areas of the polygon in Figure 6, given that the length of the side of the square in the grid is 1 cm.

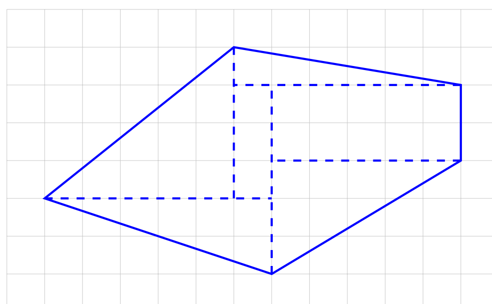


Figure 6

Solution. This figure consists of two rectangles and 4 triangles.

- central rectangle: area is 3 sq. cm.
- right rectangle: area is 10 sq. cm.
- upper left triangle: area is 10 sq. cm.
- upper right triangle: area is 3 sq. cm.
- lower left triangle: area is 6 sq. cm.
- lower left triangle: area is 7.5 sq. cm.

The sum is the area of the entire figure: 39.5 sq. cm.

To complete our list of fundamental figures, we include the circle, discussed in Chapter 5.

Example 6.

The area of a circle of radius r is $A = \pi r^2$, where π is approximately 22/7.

Example 7.

A 14 in. pizza has the same thickness as a 10 in. pizza. How many times more ingredients are there on the larger pizza?

SOLUTION. Pizzas are measured by their diameters, so the radii of the two pizzas are 7 in. and 5 in., respectively. Since the thicknesses are the same, the amount of ingredients used is proportional to the areas of the pizzas. The larger pizza has area $\pi 7^2 = 49\pi$ sq. in., and the smaller pizza has area $\pi 5^2 = 25\pi$ sq. in. The ratio of areas is

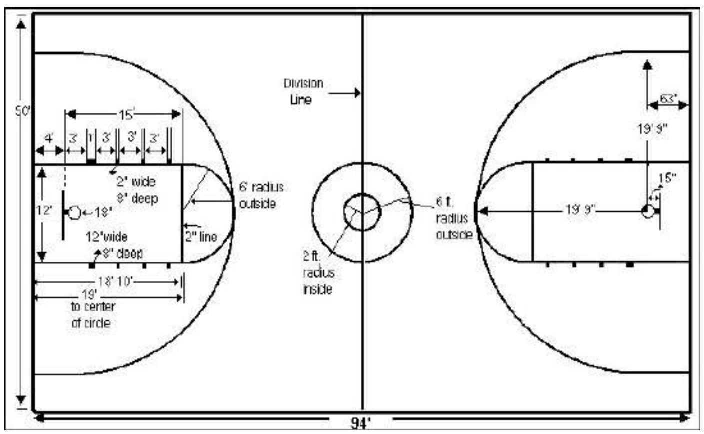
$$\frac{49\pi}{25\pi} = 1.96 \,,$$

so a 14 in. pizza has about twice the ingredients of the 10 in. one.

EXAMPLE 8.

The three-point line in basketball is approximately a semi-circle with a radius of 19 feet and 9 inches. The entire court is 50 feet by 94 feet. What is the area of the court that results in 3 points for Team A (given Team A is shooting towards its basket)?

Team A Basket



Team B Basket

SOLUTION. The three-point line is the line that separates the two-point area from the three-point area; any shot converted beyond this line counts as three points. First we decompose the area into shapes that we know well, a semi-circle and a rectangle. Then we find the area of each shape. To determine the total three-point area for Team A, we subtract the total area of the semicircle from the total area of the rectangle:

Area of rectangle = 50 ft $\times$ 94 ft = 4700 sq ft;

Area of semicircle = $\frac{1}{2}\pi r^2 = \frac{1}{2}\pi 19.75^2 = 612.71$ sq ft.

Total area that will result in 3 points for Team A;

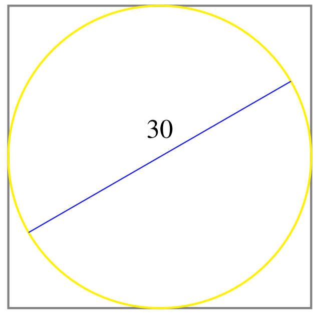
$$A_{rectangle} - A_{semicircle} = 4700 - 612.71 = 4087.29 \text{ sq. ft} \quad \text{approximately} \,.$$

Perimeter

The perimeter of any polygonal region is the sum of the lengths of its sides. So, the perimeter of a square of side length 5 in. is 20, since there are 4 sides. Contrast this to area, which is the product of the basis cimenstions; in this case 5-by-5, giving 25 sq. in.

Example 9.

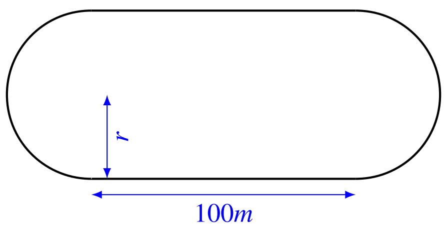
A ring made of gold that has a diameter of 30 cm is put in a silver display box so that the ring just fits. What is the length of the ring of gold? What is the length of the frame of silver?



Solution. The radius of the ring of gold is 15 cm, so the length of the ring is $2\pi \times 15 = 94.2$ cm, approximately (here we have approximated the value of π by 3.14). The perimeter of the box of side length 30 cm is 120 cm. Thus we have about four-thirds as much silver as gold.

Example 10.

Typically, a (foot) race track is formed by a rectangle with a semicircle at the short ends, so that the total distance is 400 m (about one-quarter mile), and the straight lengths (the long sides of the rectangle) are 100 m each. What is the radius of the semicircle?



Solution. The perimeter of the track is 400 m; of these there are two 100 m straightaways and two semicircles of radius r (which is the same as a full circle of radius r). So, we must have $100+100+2\pi r = 400$. Solve for r to get $r = \frac{100}{\pi} = 31.86$ m.

Example 11.

Actually, a track consists of 6 or more lanes, each of which is 1.2 m wide. What we have just calculated is the inside length of the first lane. What is the inside length of the second lane?

Solution. The radius of the semicircle at the ends formed by the inside of the second lane is 1.2 m longer than the inside length of the first lane, so is $31.86+1.2 = 33.06$ m. Then the total length around the track on the second lane is $100 + 100 + 2\pi(33.06) = 407.62$, and the second lane is thus 7.62 m longer than the first lane. In fact, each lane (on its inside) is 7.62 m longer than the preceding lane: this is why racers running a 400 m race start at intervals of separated by 7.62 m.

Contrasting and Relating Area and Perimeter

If you know the distance around a shape (its perimeter), can you determine its area? If you know the area of a shape, can you determine that distance around the shape? Does perimeter determine area? For a square, the answer is "yes": if you know the perimeter you can find the side length and from this, the area. For a circle, again the answer is "yes": if you know the perimeter (i.e., the circumference) you can find the radius and hence the area. Yet, they these are different types of measurements, and they are expressed using different units, and are calculated in two different ways. Perimeter is measured in units of length, and area is measured in square units of length. Furthermore, perimeter is calculated by adding lengths, and area is calculated by multiplying lengths appropriately. This is strictly true for rectangles: perimeter is the sum of all of the sides, and area is found by multiplying the two side lengths.

Example 12.

Calculate the area and the perimeter of a rectangle of side lengths 6 ft. and 4 ft., and another rectangle of side lengths 8 ft. and 2 ft. See Figure 7.

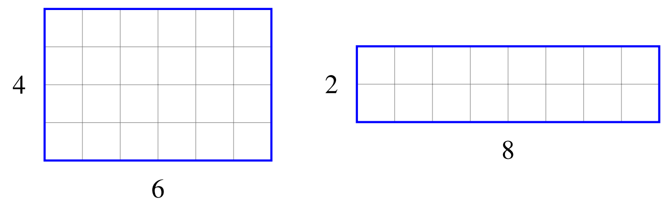


Figure 7

The areas are 24 sq. ft. and 16 sq. ft., but the perimeters of both rectangles are the same: 20 ft.

If we calculate the areas of many rectangles of the same perimeter, we see that the if the side lengths are closer in measure then the area of the rectangle is larger. That can be shown graphically for the two rectangles of Example 12. In Figure 8, notice that as the longer side of the rectangle is made shorter, the shorter side of the rectangle becomes longer to preserve the perimeter. The effect is to lose some area at the short end but gain more area on the long side of the rectangle for an overall increase in area. If we continue to decrease the length of the longer side and increase the length of the shorter side, we see the same phenomenon until we reach the rectangle with all sides the same length, 5. This gives strong evidence that, among all rectangles of given perimeter, the square has the largest area. Students should experiment with this statement with a variety of rectangles.

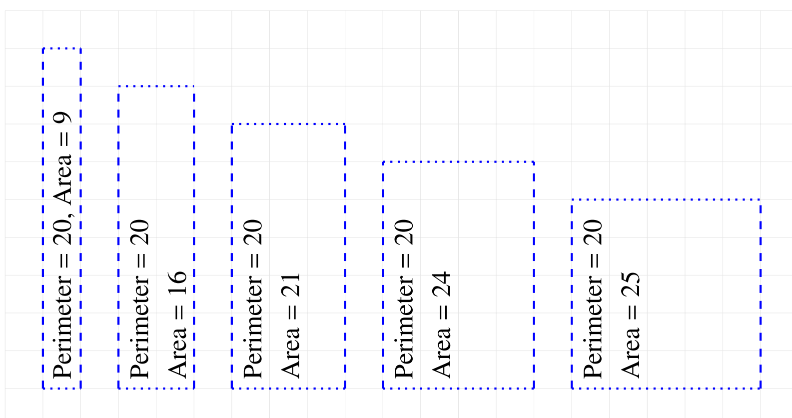


Figure 8

Example 13.

Let's follow through on this with another example: Among all rectangles of perimeter 50 units, find the rectangle with the largest area. If the long side of the rectangle is 15 units, the shorter side has length 10 units, and the area is 150 square units. If the long side has length 20 units, the short side has length 5 units and the area is 100 square units. Here is a table of values of the area of a rectangle with a given long side:

Long side length (in units)	24	23	22	21	20	19	18	17	16	15	14	13	12
Area (in square units)	24	46	66	84	100	114	126	136	144	150	154	156	156

We have stopped the computation at the length 12 units, for from that point on this is not the length of the "long side." In fact, the square with the perimeter 50 units has side length 12.5 units and area 156.25 square units, and that is the rectangle with perimeter 50 units and greatest area. It is easier and more fun to experiment with this at websites set up for this purpose. For example,

`http://www.mathopenref.com/triangleareaperim.html`

is such a site in which area and perimeter are interactively explored with triangles.

Section 8.2: 2D Plane Sections of 3D Figures, 3D Measurement

Describe the two-dimensional figures that result from slicing three-dimensional figures, as in plane sections of right rectangular prisms and right rectangular pyramids. 7.G. 3.

Solve real-world and mathematical problems involving area, volume and surface area of two- and three- dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms. 7.G.6.

Now we turn to shapes in 3D and how to visualize them using our knowledge about 2D figures. Students begin by examining plane sections of 3D figures. A *plane section* of a solid in 3D is the 2 dimensional figure one gets by slicing the solid along some plane in space. In Figure 9a we show a plane section of a bagel.

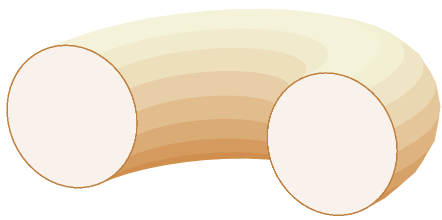


Figure 9a

This is not the usual section of a bagel, which will be along the plane of its major diameter. In that case, we get an image like this:

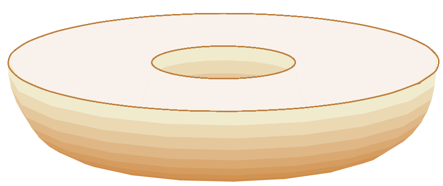


Figure 9b

In common language, a plane section is also called *cross section*. Often the name cross section is reserved for a section of a 3D object that is parallel to a particular plane of symmetry of the object, or perpendicular to a particular line of symmetry. For example, one line of symmetry for a cube is a line joining the centers of opposite faces, and a cross section perpendicular to that line is a square.

The emphases of this section are:

1. Describe the different ways to slice a 3D figure.

2. Describe the different 2D cross-sections that will result depending on how you slice the 3D figure.
3. Solve real-world and mathematical problems involving volume and surface area of three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

Because one of the purposes of this section is to help students visualize and draw (or otherwise represent) three-dimensional figures, a brief review of some geometric terms and definitions is appropriate.A closed, connected shape in space whose outer surfaces consist of polygons such as triangles, squares, or pentagons is called a *polyhedron* (*polyhedra* is the plural). The polygons that make up the outer surface of the polyhedron are called the faces of the polyhedron. The line segment where two faces come together is called an edge of the polyhedron. A corner point where several faces come together is called a vertex (vertices is the plural) or corner of the polyhedron. The name polyhedron comes from the Greek; *poly* meaning "many" and *hedron* meaning "face," so polyhedron literally means "many faces." Figure 11 depicts a typical polyhedron. This polyhedron has 9 vertices, 14 edges and 7 faces.

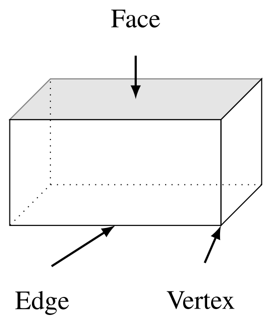


Figure 10

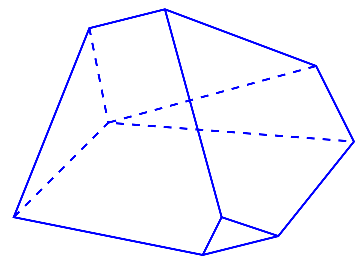


Figure 11

General Prisms

Start with two planes in space that are parallel. For any polygonal figure in one of the planes, sweep it through space in a direction perpendicular to the starting plane until we reach the ending plane. The resulting 3D figure is called a *prism*. In Figure 12 are three 3D figures obtained by sweeping a figure in the bottom plane out to the top plane. The first figure is a representation of a general prism. The second is called a *circular cylinder* (or, commonly, a *cylinder*). Since it is formed by sweeping out a circle and not a polygon, it is not a prism. The last is a prism, called the *triangular prism* or *wedge* since it is formed by sweeping out a triangle.

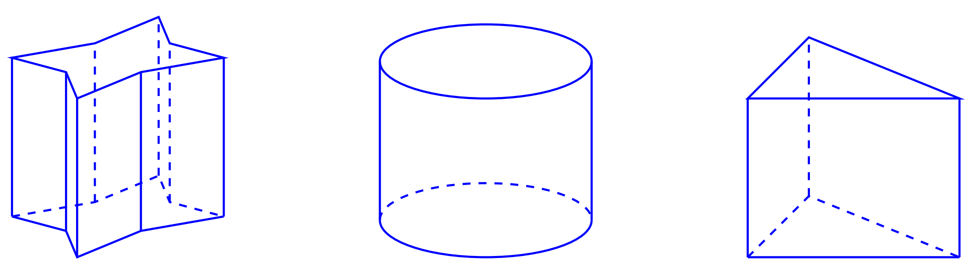


Figure 12

Technically, these should be called *right prisms*, with *right* specifying that these figures have been drawn out in a direction perpendicular to the plane of the start figure. Since, in 7th grade, we consider only this case, we will not use the adjective *right*. Notice also that any section of such a solid by a plane parallel to the original planes is a copy of the original planar figure. In Figure 13 there are three more solids of interest that are specific prisms:

Note that a cube is a special kind of rectangular prism, one in which all edges are of the same length. For a prism, we shall refer to the planar figure from which it is drawn out as the *base*, and the distance it is drawn out as its

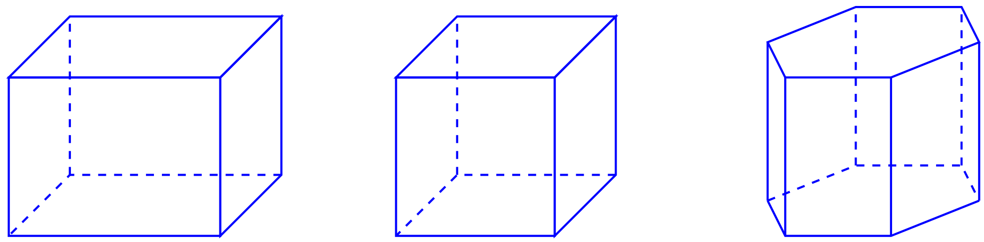


Figure 13

height. In the first set of figures and the hexagonal prism, there is only one face that qualifies as the base; however, the rectangular prism can be realized as drawing out of any one of its faces, so the word "base" could be attributed to any one of its faces. In any problem, pick a face to be the base if it makes the problem simpler; otherwise it doesn't matter.

General Pyramids

Start a plane in three-dimensions and a point A not on the plane, and a polygonal figure F on the plane. Attach all points on A to F by line segments. The resulting 3D figure is the *pyramid* w ith *base* F and *apex* A. For a polygonal figure in one of the planes, sweep it through space in a direction perpendicular to the starting plane until we reach the ending plane. The resulting 3D figure is called a *pyramid*. The first solid in Figure 14 is a *rectangular pyramid* because its base is a rectangle (probably a square) and the last solid is a *triangular pyramid* because its base is a triangle. This figure is also called a tetrahedron because it consists of four faces, all triangles. If the faces are all equilateral triangles, it is a *regular tetrahedron*. The middle figure, the *circular cone* also consists of all lines from a planar figure to a point not on the plane, but since the starting figure is not a polygon, it is not a pyramid.

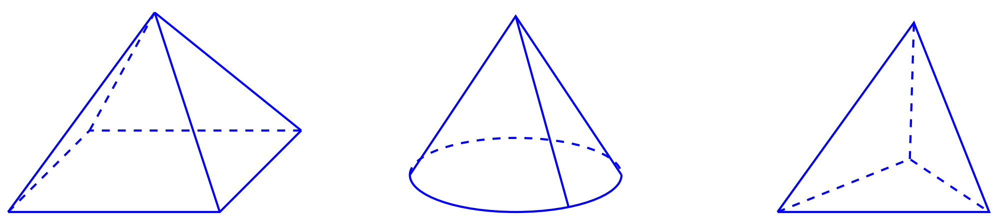


Figure 14

We shall be considering pyramids with the property that the line from the apex to the center of the base is perpendicular to the base (technically, these are called *right* pyramids). Note that sections of the pyramid by a plane parallel to the base produces a figure that is a scaled version of the base, with the scale factor reducing as we move toward the apex. In our figure for the pyramid, the base appears to be a square, but it need not be. The great pyramids in Egypt are all built above a square base, whereas many Mayan pyramids (in Mexico) have rectangular (non-square) bases.

2D Plane Sections of 3D Figures

We now focus on two-dimensional aspects of solid shapes: plane sections. Plane sections provide two-dimensional information about the inside of a shape. Thinking about cross-sections can help us recognize that solid shapes have an interior in addition to an outer surface. Even as we study volume in section 8.2, we can think of the volume of a prism as decomposed into layers, where each of these layers are much like thickened cross-sections. As mentioned earlier, when the sectioning plane is perpendicular to a particular axis of the figure, we will use the term *cross section*; it turns out that such sections are particularly important for prisms and cones.

The intent in the first example following is to have students explore all possible sections of a cube by a plane. Our text goes through this systematically, but this exposition should be considered as a chart of possible end results, and *n*ot to be provided to students with a guide. The following examples extend this to other figures, going beyond the intent of the standard, and thus are designated as Extensions.

EXAMPLE 14.

What shapes can be created by one slice through a cube? Look for these possibilities:

- **a.** a square
- **b.** an equilateral triangle
- **c.** a rectangle that is not a square
- **d.** a triangle that is not equilateral
- **e.** a pentagon
- **f.** a hexagon
- **g.** a parallelogram that is not a rectangle
- **h.** an octagon
- **i.** a circle

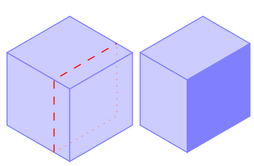
SOLUTION. First, a comment about plane sections of a general polyhedron. Each side of a plane section comes from cutting through a face on the polyhedron. When two planes in space intersect, they intersect in a line. Thus the edges of a plane section all have to be line segments; no curved edges are possible. So the answer to part i) is "no, it is impossible to get a circle" because any plane section of a polyhedron has to be a polygon. Furthermore, since a plane section intersects each face in just one line segment, the plane section cannot have more edges than the polyhedron has faces.

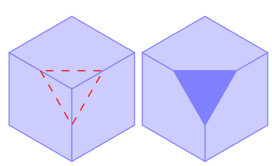
Since a cube has 6 faces, a plane section can have at most 6 sides; so this answers part h in the negative: no octagons (and in fact, no seven-sided polygons either).

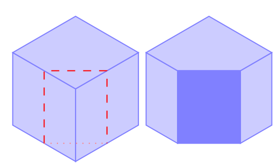
- **a.** A square cross section can be created by slicing the cube by a plane parallel to one of its sides. This is the only way to get a square as a section of the cube; furthermore they are all of the same size as any face.

- **b.** An equilateral triangle can be obtained by a plane section by cutting the cube by a plane that is perpendicular to the diagonal joining two opposed vertices of the cube. The largest such triangle is obtained when the plane of the section includes three vertices of the cube.

- **c.** A plane that is perpendicular to a face, but not parallel to any face will cut the cube in a rectangle that is not a square. Every such rectangle has area less than that of a face.


a.

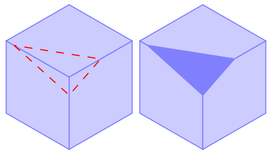

b.


c.

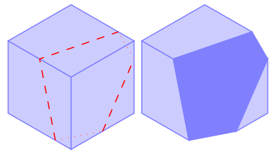
- **d.** Pick a vertex, let's say A, and consider the three edges meeting at the vertex. Construct a plane that contains a point near a vertex (other than vertex A) on one of the three edges, a point in the middle of another one of the edges, and a third point that is neither in the middle nor coinciding with the first point. Slicing the cube with this plane creates a cross section that is a triangle, but not an equilateral triangle; it is a scalene triangle. Notice that if any two selected points are equidistant from the original vertex, the cross section would be an isosceles triangle.

e. To get a pentagon, slice with a plane going through five of the six faces of the cube.

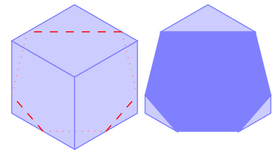
f. To get a hexagon, slice with a plane going through all six faces of the cube.



d.

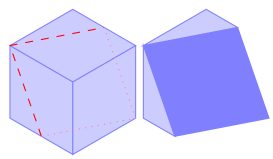


e.



f.

g. To create a non-rectangular parallelogram, slice with a plane from the top face to the bottom. The slice cannot be parallel to any side of the top face, and the slice must not be vertical. This allows the cut to form no 90° in angles. One example is to cut through the top face at a corner and a midpoint of a non-adjacent side, and cut to a different corner and midpoint in the bottom face.



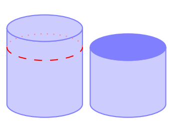
g.

Extension

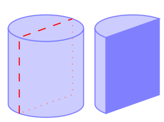
Example 15.

What shapes can be created by one slice through a circular cylinder?

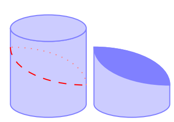
Solution. Possible plane sections are: a circle (cut parallel to the base), a rectangle (cut perpendicular to the base), ellipse, or a cutoff ellipse.



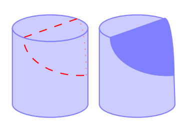
Circle



Rectangle



Ellipse



Cutoff Ellipse

Example 16.

What shapes can be created by a slice through a square pyramid?

Solution. Refer to Figure 15.

a. If the pyramid is cut with a plane parallel to the base, the intersection of the pyramid and the plane is a square cross section.

b. If the pyramid is cut with a plane passing through the top vertex and perpendicular to the base, the intersection of the pyramid and the plane is a triangular cross-section.

c. If the pyramid is cut with a plane perpendicular to the base and parallel to one of the edges of the base, but not through the top vertex, the intersection of the pyramid and the plane is an isosceles trapezoidal cross-section.

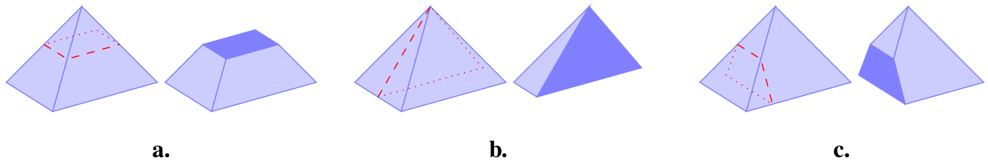


Figure 15

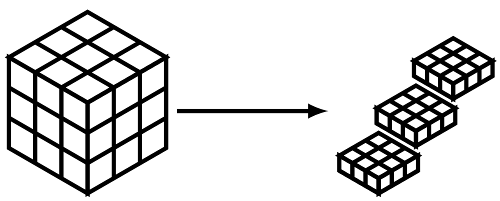
If the plane of the section is neither perpendicular to the base, nor parallel to an edge, can we find other polygons?

End Extension

Volume

The volume of a solid shape is a measure of how much three-dimensional space the shape takes up. What does it mean to say that the volume of a solid shape is 27 cubic centimeters? It means that the solid shape could be made (without leaving any gaps) with a total of 27 1 cm × 1 cm × 1 cm cubes, allowing cubes to be cut apart and pieces moved if necessary. If we thought of a box as subdivided into layers, and each layer as made up of 1unit × 1unit × 1unit cubes, and each small cube has a volume 1 cubic unit, then the volume of the whole box (in cubic units) is the sum of the volumes of the cubes, which is just the number of cubes.

Consider the popular puzzle called *Rubik's Cube*.



When you think about a traditional Rubik's Cube there are three layers. In each of these layers are nine smaller cubes. When you multiply three by nine you get twenty-seven.

The most basic way to determine the volume of a solid shape is to make the shape out of cubes (filling the inside completely) and to count how many cubes it took. Now not every solid shape is made out of cubes, but if we take the unit cube small enough, this method will produce a good approximation of the volume. Although primitive, this method is important, because it relies directly on the definition of volume and therefore emphasizes the meaning of volume.

A way to find the volume of a solid shape is to understand how it can be developed out of two dimensional figures. For example, as we saw above, a cube can be viewed as a stack of squares, all of which have the same side lengths. We now use this idea to calculate volume. Let's look at prisms and cones, as they are solid shapes created by drawing out a planar polygon. We focus on sections by planes parallel to the base of the figure. If the sections are all of the same size and shape, we have a *prism* and if the they are scalings of the original we have a *cone*. Let's look at these two types more closely.

Volume of a Prism

A prism is described as the solid formed by drawing out a figure on a plane for a specified distance along parallel lines emanating from the plane. If the direction is perpendicular to the starting plane, it is called a *right prism*. We shall focus on right prisms. We want to establish this formula to compute the volume of prisms:

- The volume of a prism is the product of the height by the area of the base. That is, if the area of the base is B and the height is h, volume is $V = Bh$.

Of the three prisms illustrated in Figure 13, the first two can be viewed as having been formed by drawing out any one of its faces in the perpendicular direction. In calculating area we will make a choice of face to serve as base; in most cases it doesn't matter, and often the context directs us to a proper choice of base.

Example 17.

Figure 16

The National Press Building on Fourteenth Street and Avenue F is 14 stories high, with 12 feet to each story. It has 150 feet of frontage on 14th St, and 200 feet on Ave F. The building has the shape of a rectangular prism. What is its volume?

We view the building as formed by drawing a 150×200 rectangle upwards for 14 stories. Now the area of the base is $150 \times 200 = 30,000$ sq. ft. Since each story is 12 feet high, the volume of each story is $12 \times 30,000 = 360,000$ cu. ft., and as the building is made up of a stack of 14 stories identical to the first one, the total volume is $14 \times 360,000 = 5,040,000$ cubic feet.

Next we turn to prisms that do not have a rectangular base to see that the above assertion about volume is true. Start at the planar figure at the base of a prism: its area is approximated by covering the figure with a grid of squares, and counting the number of squares inside the figure. The finer the grid the better the approximation. Figure 17 is of a prism over a trapezoidal base B. If we cover B with a grid, we can draw out that grid along the

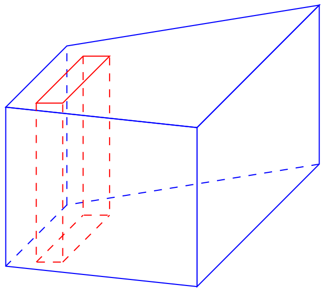


Figure 17

parallel lines, getting a decomposition of that region into rectangular prisms, the base of each is a rectangle in the planar grid. Our figure shows a typical such rectangle. By filling the solid with rectangular prisms of this type, *all with the same height*, then adding together the volumes of all the rectangular prisms inside the prism, we get the formula "Area = base×height" for this approximation. As the grid gets finer and finer, the approximations get better and better, but the formula remains the same. So the statement $V = Bh$ is confirmed for the general prism.

Example 18.

The Pentagon, the headquarters of the U.S. Department of Defense, is a regular five-sided figure with a total of 6.5 million square feet of floor space on seven levels, two of which are underground. The side length of the interior central plaza is about 1/3 the side length of the building.

a. What is the measure of the *footprint* of the Pentagon? The footprint is the total area occupied by the building together with the central plaza.

b. What is the area of the central plaza?

c. There are 11 feet of elevation between floors of the Pentagon. What is the total volume of the above-ground building?

Solution.

a. The image shows the Pentagon to be a prism - in the sense that all floors are of the same shape and size; indeed all sections by planes parallel to the ground are of the same shape and size. Thus each floor of the building comprises 1/7 of 6.5 million sq. ft., or 928,571 sq. ft. But this is the area of the base floor of the building, not the footprint, which includes the central plaza, We are told that the length of a side of the plaza is one-third the side length of the building. Since the plaza and the building have the the same shape, that tells us that the footprint of the plaza is a downscaling of the footprint of the entire Pentagon by a linear scale factor of 1/3. Since area scales by the square of the linear scale factor, we conclude that the area of the plaza is 1/9th of the are of the footprint. Thus the area of the floor of the building, 928,000 square feet is 8/9 of the area of the footprint. The answer then, to a) is that the area of the footprint is $\frac{9}{8}(928,000) = 1,044,000$ sq. ft.

b. The plaza is 1/9 of the footprint, so its area is $\frac{1}{9}(1,044,000) = 116,000$ sq. ft.

c. The reason this figure (the volume of the building) is interesting is to estimate the cost of heating the building in winter, and air-conditioning it in summer. So, now we are interested only in the volume of the building that is above ground. Since there are 5 stories above ground, each of height 11 feet, the bullding stands 55 feet high. The area of the base is 928,000 sq. ft., so the volume of the building above ground is $55 \times 928,000 = 51,040,000$ cu. ft.

If the prism is not a right prism, as in Figure 28, can we still find its volume?

We can view the figure as a collection of copies of the base figure, but this time, not directly on top of one another, but each moved somewhat askew, as on the right in Figure 19:

The volume of the stack of boxes together is the same in both figures. If we take the height of all the boxes small enough, we get a very good approximation of the volume of the figure above.

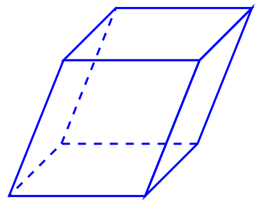


Figure 18

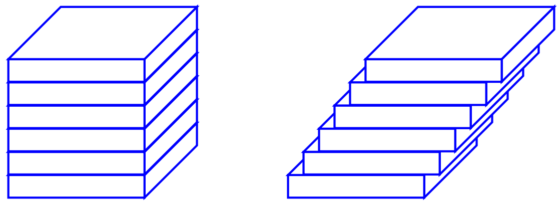


Figure 19

In this example, we see an application of

- **Cavalieri's principle**: Suppose that we stand two figures side by side. Suppose that every horizontal slice through the two figures gives two planar figures of the same area. Then the volume of the two solid figures is the same.

Notice that we do not require that the figures have the same size and shape, only that they have the same area. But, since figures of the same size and shape have the same area, Cavalieri's principle applies to all prisms.

Volume of a Pyramid

A pyramid has been described as the solid formed by the aggregate of line segments joining points on a given polygonal figure (the base) in a plane to a single point A (the apex).

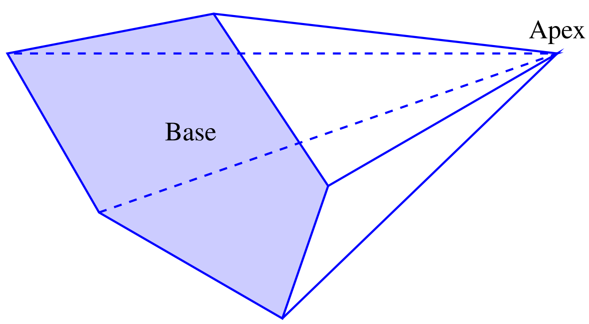


Figure 20

If the base has a center, and the line from the center to the apex is perpendicular to the base, we call the solid a *right pyramid*. The figure above is a generic pyramid whose base is a polygon with five sides. The shapes in Figure 14 are all right pyramids.

The *height* of a pyramid is the distance from the apex to the plane of the base. The formula for the volume of a pyramid is:

- The volume of a pyramid is one-third the product of the height by the area of the base. That is, if the area of the base is B and the height is h, volume is $V = \frac{1}{3}Bh$.

This fact was discovered by the the ancient Greeks by direct experimentation with a variety of pyramids. Here is a statement of their conclusion: Start with a pyramid whose apex lies somewhere over the base. The *circumscribed prism* is the prism of the same height with the same base. Figure 21 depicts this pair for a right pyramid over a rectangle.

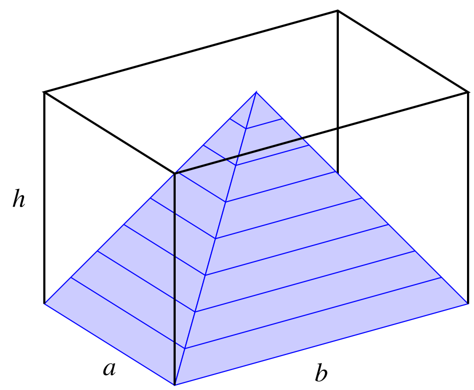


Figure 21

What was observed was that the volume of the pyramid is 1/3 that of its its circumscribed prism. This can be shown by creating containers with these shapes and comparing the volume of water that fills the objects. It takes exactly three fillings of the pyramid to fill its circumscribed prism. The Greeks went further with a specific pyramid, the tetrahedron. Three models of the tetrahedron can be put together to completely fill the circuscribed cube.

By extension, the Greeks concluded that the volume of a circular cone is 1/3 the volume of its circumscribed cylinder (students will return to this in 8th grade.

Still, this is not a geometric argument as the ancient Greeks would have wanted it to be. Even today, we have no satisfactory way to visualize putting three circular cones of the same size and shape into the circumscribed cylinder.

Extension. Here is a construction that comes close. Start with the cone whose base is a square, and whose height is equal to the side length of the base. Furthermore, put the apex of the cone directly over one of the vertices of the base square. Construct the cube that is its circumscribed prism. See Figure 22.

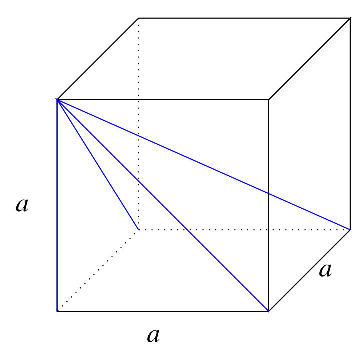


Figure 22

Make a box of the dimensions of circumscribed cube, and a pyramid as indicated by the figure. The sequence of images in Figure 23 show how to fill the cube with three copies of the cone.

This construction and the physical measurements of volume with water all support the conclusion that the volume

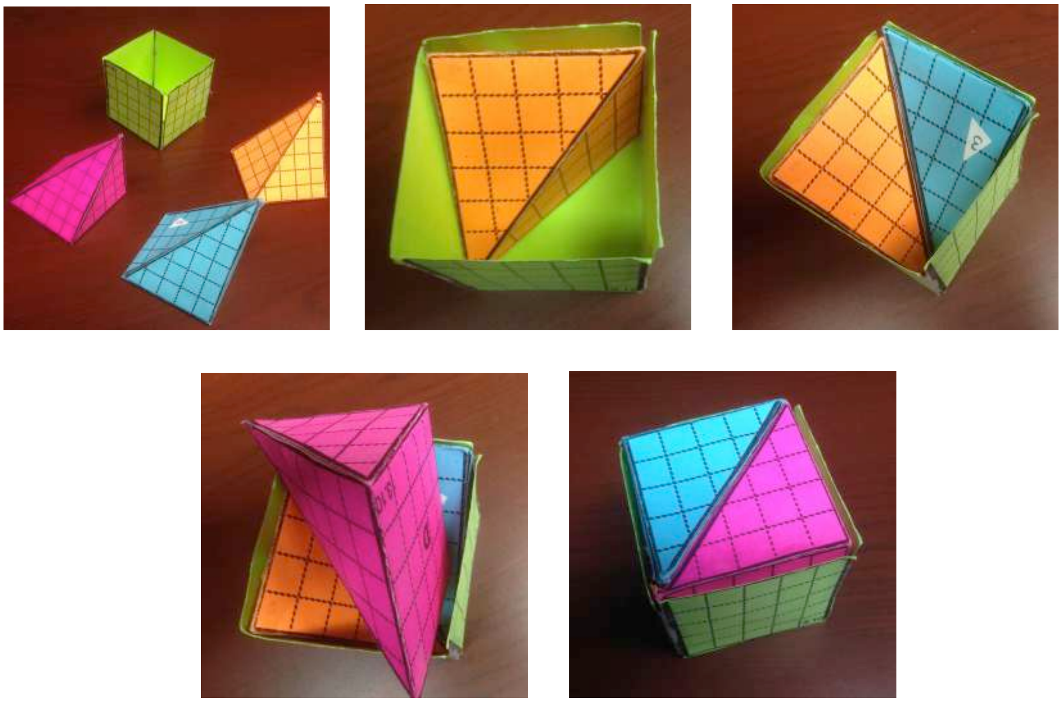


Figure 23. Thanks to the creator, Marilyn Keir

of a cone is 1/3 the volume of its circumscribed prism. But, for the Greeks, this did not explain why? Why 1/3? When we move from a right triangle to its circumscribed rectangle, we have a factor of 1/2; moving from two dimensions to three, we introduce a factor of 1/3. This seemingly strange geometric fact, together with the unconstructive nature of the parallel postulate, worried mathematicians strictly adherent to the principle of logic for almost 2 thousand years.Today the factor 1/3 is easily understood thanks to a calculation by Cavalieri of an area bounded by parabola, and later incorporated as one of the building stones of the Calculus.

End Extension

Example 19.

The 7th graders at Albert R. Lyman Middle School were helping to renovate a playground for the kindergartners at the nearby Blanding Elementary School. Blanding City regulations require that the sand underneath the swings be at least 15 inches deep. The sand under both swing sets was only 12 inches deep when they started. The rectangular area under the small swing set measures 9 feet by 12 feet and required 40 bags of sand to increase the depth of 3 inches. How many bags of sand will the students need to cover the rectangular area under the large swing set if it is 1.5 times as long and 1.5 times as wide as the area under the small swing set?

Solution. There are many different ways to approach and solve this problem. Let's consider three approaches that student's might take with respect to volume, scale factor or unit rate.

Solution 1 (volume): 3 inches is 1/4=0.25 feet, so the volume of sand that was used is $0.25\times 9\times 12 = 27$ cubic feet. The amount of sand needed for an area that is 1.5 times as long and 1.5 times as wide would be $0.25 \times (1.5 \times 9) \times (1.5 \times 12) = 60.75$ cubic feet. We know that 40 bags covers 27 cubic feet. Since the amount of sand for the large swing set is $60.75 \div 27 = 2.25$ times as large, they will need 2.25 times as many bags. Since $2.25 \times 40 = 90$, they will need 90 bags of sand for the large swing set.

Solution 2 (scale factor): Since we have to multiply both the length and the width by 1.5, the area that needs to be covered is $1.5^2 = 2.25$ times as large. Since the depth of sand is the same, the amount of sand needed for the large swing set is 2.25 times as much as is needed for the small swing set, and they will need 2.25 times as many bags. Since $2.25 \times 40 = 90$, they will needs 90 bags of sand for the large swing set.

SOLUTION 3 (UNIT RATE): The area covered under the small swing set is $9 \times 12 = 108$ square feet. Since the depth is the same everywhere, and we know that 40 bags covers 108 square feet, they can cover $108 \div 40 = 2.7$ square feet per bag. The area they need to cover under the large swing set is $1.5^2 = 2.25$ times as big as the area under the small swing set, which is $2.25 \times 108 = 243$ square feet. If we divide the number of square feet we need to cover by the area covered per bag, we will get the total number of bags we need; $243 \div 2.7 = 90$. So they will need 90 bags of sand for the large swing set.

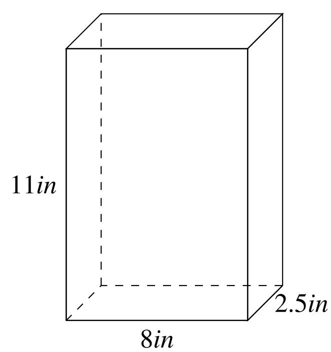
Contrasting and Relating Volume and Surface Area

In sixth grade, using the idea of nets, students worked out strategies to find the area of the surface of a polygonal figure in space: adding up the surface areas of all the faces. The decomposition into nets provided a way to organize this computation. In particular, a rectangular prism of side lengths 2, 3 and 5 units has six faces, 2 each of dimensions 2×3 sq. un., 2×5 sq. un., and 3×5 sq. un., so the surface area is $2(2\times3+2\times5+3\times5) = 2(6+10+15) = 62$ sq. un. Here we want to understand the difference between volume and surface area and the relation between them.

First of all, why is surface area important? A painter would be interested in the surface area of a room, rather than volume. Chemotherapy treatment of cancer takes place through the surface of the growth, so the surface area of the cancer is a more important parameter than its volume.

EXAMPLE 20.

Manufacturers sell breakfast cereals by volume or weight (usually 25 oz.) but determine the size of containers on economic and aesthetic grounds. Here is a typical example: what is the area of the material necessary to cover the surface of the box?



SOLUTION. A rectangular prism has 6 faces, identical in opposing pairs. The dimensions of the faces in this case are 11×8, 11×2.5, and 8×2.5 sq. in. Doing the multiplication we have two faces of 88 sq. in., 2 faces of 27.5 sq. in. and 2 faces of 20 sq. in. Therefore the total area is $2(88+27.5+20) = 135.5$ sq. in.

EXAMPLE 21.

In the movie *Despicable Me*, an inflatable model of The Great Pyramid of Giza in Egypt was created by Vector to trick people into thinking that the actual pyramid had not been stolen. When inflated, the false Great Pyramid was 225 m high, with a slant height of 230.5 m for any one of the triangle faces (by *slant* height, we mean the distance from its base to the apex of the pyramid in the plane of the triangle). The base a square with each side 100 m in length. How much material did Vector need in order to re-create The Great Pyramid of Giza?

Solution.

$$\text{Area}_{triangle} = \frac{1}{2}(100 \times 230.5) = 11,525 \text{ sq. m}$$

$$\text{Area}_{base} = (100 \times 100) = 10,000 \text{ sq. m}$$

$$\text{Total Surface Area} = (10,000) + 4(11,525) = 56,100 \text{ sq. m.}$$

Example 22.

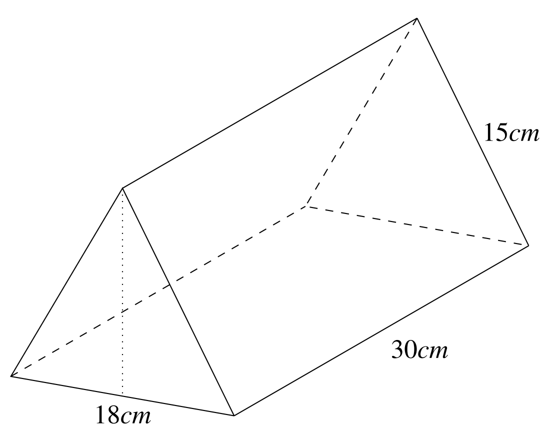
A baker creates fantastic cupcakes that can each be comfortably enclosed in a cube of side length 2 in. She wants to deliver these cupcakes to stores in batches of 75 in a large box with a square base that has 3 layers of cupcakes. What is the volume of the box, and what is its surface area?

Solution. Since the 75 cupcakes are placed in the box in three square layers, each layer has 25 cupcakes in a 5×5 array. Since each cupcake a cube of side length 2 in, the square base of the box has side length 10 in. Each cupcake cube is 2 in high, and there are three layers, so that box is 6 in high. The volume of the box is the product of its side lengths: $V = 10 \times 10 \times 6 = 600$ cu. in. The area of the bottom is 100 sq. in. and the area of a side is 60 sq. in. Since the top has the same area as the bottom, and there are 4 sides, the surface area is $A = 2(100) + 4(60) = 440$ sq. in.

In packaging, one must know the edge dimensions, the surface area and the volume; however, some of these are more significant that others. For example, if we are packaging heavy objects, the volume is most important, but if we are wrapping it with an expensive paper, surface area may be paramount, and if we tie it up in gold ribbon, the edge dimensions count the most.

Example 23.

Mr. Brewer purchased a box of his favorite chocolates. The box is in the shape of a prism whose base is an isosceles triangule (see the diagram below), and the edge dimensions of the prism are as shown. If the volume of the box is 3,240 cubic centimeters, what is the height of the triangular face of the box? How much packaging material was used to construct the package



Solution. Volume is found by multiplying the area of the base (isosceles triangle) by the length of the prism: $V = Bl$. Here we are given the dimensions $V = 3240$ cu. cm., $l = 30$ cm., so we must have $B = 108$ sq. cm. Now the area of a triangle is one-half the product of the base and the height, and here we know that the base is 18 cm, so we have $\frac{1}{2}(18)h = 108$, giving us $h = 216/18 = 12$ cm.

The problem also asks for the surface area of the package. Find the area of each face and add:

2 triangular bases, each of which is 108 sq. cm. contribute 216 sq. cm.;

2 rectangular faces (the side faces in the diagram) that are 15×30 each contribute 450 sq. cm.;

One 30 cm. $\times$ 18 rcm. rectangular bottom contributes 540 sq. cm.

The total produces $216 + 2 \times 450 + 540 = 1656$ sq. cm.

Is there a relationship between surface area and volume? Can rectangular prisms with different dimensions have the same volume? Do rectangular prisms with same volume have the same surface area?

Example 24.

For shipping purposes, cubes of fudge need to be packaged in boxes that are rectangular prisms. Knowing the company only sells their fudge cubes in groups of 24, what are the possible dimensions for the boxes?

Solution. . Begin by recording the information in table form.

Length	Width	Height	Volume	Surface Area
1	1	24	24 cu. cm	98 sq. cm
2	1	12	24 cu. cm	76 sq. cm
3	1	8	24 cu. cm	70 sq. cm
4	1	6	24 cu. cm	68 sq. cm
2	2	6	24 cu. cm	56 sq. cm
2	3	4	24 cu. cm	52 sq. cm

Which of the packages requires the least and the greatest amount of material and why would it be important? The package that requires the least amount of material is the $2 \times 3 \times 4$ package and the package that requires the greatest amount of material is the $1 \times 1 \times 24$. Why is the amount of material important to a company? Because material costs money and the more material would equate to a higher cost.

What conclusion can we make about the shape of the package with the smallest and greatest surface area and what would you recommend to the fudge company? Notice that the shape of the package with the smallest surface area is the package that most closely resembles a cube and the package with the greatest surface area is the package that is least like a cube. Recommending the $2 \times 3 \times 4$ package would mean that the company would save money by using the least amount of material possible.

Looking at the table, what is the relationship between surface area and volume? Notice that surface area decreases as the rectangular prisms move closer to the shape of the cube. Another key revelation is that rectangular prisms with different dimensions can have the same volume. Lastly, rectangular prisms with the same volume can have different surface area. For example, the number of exposed faces of each unit cube is different for each prism. The $1 \times 1 \times 24$ prism has 22 cubes with four exposed faces and the two end cubes have five exposed faces. The $2 \times 3 \times 4$ has 8 cubes with three exposed faces (the corners), 12 cubes with two exposed faces, and 4 cubes with one exposed face.

Example 25.

I was planning to send a gadget that is 8 in. by 12 in. by 14 in. to my Aunt Sarah.

a. What is the volume of the box I would need?

b. As it turns out, that gadget is no longer available, but I can send her a scale-reduced model of the gadget, reduced to 60% in every linear dimension. Now what volume box do I need?

c. Looking further on the internet I find an opportunity to purchase and send a similar object that is 4 in longer than the one I was considering. However, I am not sure which dimension is the length (Note, length is typically the longest dimension).What are the possible volumes of these packages? What are the possible surface areas of these packages?

SOLUTION.

a. The volume is $8 \times 12 \times 14 = 1344$ cu. in. A cubic foot is the same as 1728 cu. in., so the package contains $1344/1728 = .78$ cu. ft.

b. This reduced-size gadget has been reduced by 60% of its original value in every linear dimension. Thus the new volume is $(0.60)^3 \times 0.78 = 0.16848$ cu.ft., or about a sixth of a cubic foot.

c. The three possibilities are: first, $(8 + 4) \times 12 \times 14 = 2016$ cu. in.; second, $8 \times 16 \times 14 = 1792$ cu. in.: and the third is $8 \times 12 \times 18 = 1728$ cu. in.

Extension

EXAMPLE 26.

For a rectangular prism, if we know the area of each its faces, do we know its volume?

SOLUTION. The answer is "yes," and there are many ways to show this. First, we should suspect that the answer is "yes" since it takes 3 numbers (the lengths of the sides) to determine the volume, and if we are given the areas of the faces, we have 3 numbers, so that should suffice. This suggests that, when we are given the face areas, we can solve for the edge lengths.This is in fact true: Let a, b, c be the side lengths, and A, B, C the given face areas. We can relate these areas to the side lengths by the equations:

$$A = bc \quad B = ac \quad C = ab\ .$$

If we multiply the left sides together we get ABC, and if we multiply the right sides together we get $a^2b^2c^2 = V^2$, since the volume $V = abc$. Thus V is the square root of the product of the sides.