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Transforming Mathematics Education

SECONDARY
MATH TWO

An Integrated Approach

MODULE 8

Circles & Other Conics

MODULE 8 - TABLE OF CONTENTS

CIRCLES AND OTHER CONICS

8.1 Circling Triangles (Or Triangulating Circles)

A Develop Understanding Task
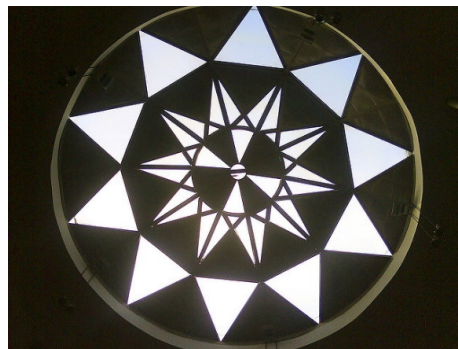
Using the corner of a piece of colored paper and a ruler, cut a right triangle with a 6" hypotenuse, like so:

Use this triangle as a pattern to cut three more just like it, so that you have a total of four congruent triangles.

1. Choose one of the legs of the first triangle and label it x and label the other leg y. What is the relationship between the three sides of the triangle?

2. When you are told to do so, take your triangles up to the board and place each of them on the coordinate axis like this:

   Mark the point at the end of each hypotenuse with a pin.

1
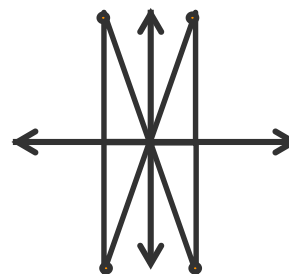
3. What shape is formed by the pins after the class has posted all of their triangles? Why would this construction create this shape?

4. What are the coordinates of the pin that you placed in:
   a. the first quadrant?
   b. the second quadrant?
   c. the third quadrant?
   d. the fourth quadrant?

5. Now that the triangles have been placed on the coordinate plane, some of your triangles have sides that are of length –x or –y. Is the relationship $x^2 + y^2 = 6^2$ still true for these triangles? Why or why not?

6. What would be the equation of the graph that is the set on all points that are 6" away from the origin?

7. Is the point (0, -6) on the graph? How about the point (3, 5.193)? How can you tell?

8. If the graph is translated 3 units to the right and 2 units up, what would be the equation of the new graph? Explain how you found the equation.

READY

Topic: Factoring special products

Factor the following as the difference of 2 squares or as a perfect square trinomial. Do not factor if they are neither.

1. $b^2 - 49$

2. $b^2 - 2b + 1$

3. $b^2 + 10b + 25$

4. $x^2 - y^2$

5. $x^2 - 2xy + y^2$

6. $25x^2 - 49y^2$

7. $36x^2 + 60xy + 25y^2$

8. $81a^2 - 16d^2$

9. $144x^2 - 312xy + 169y^2$

SET

Topic: Writing the equations of circles

Write the equation of each circle centered at the origin.

10.
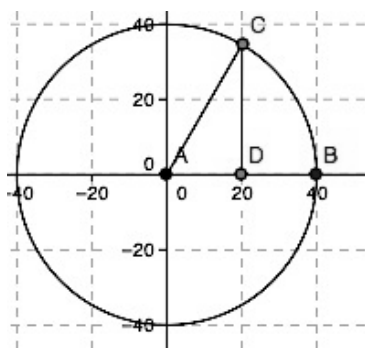
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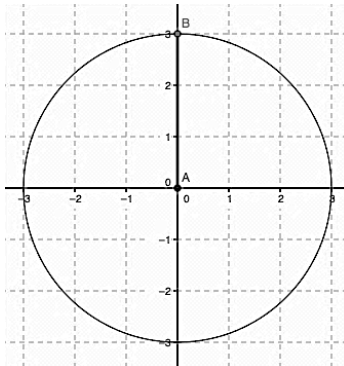
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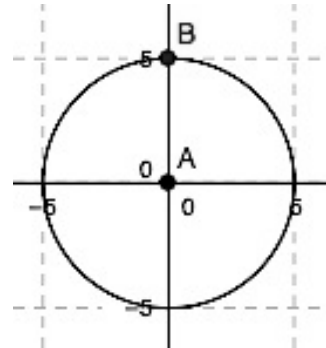
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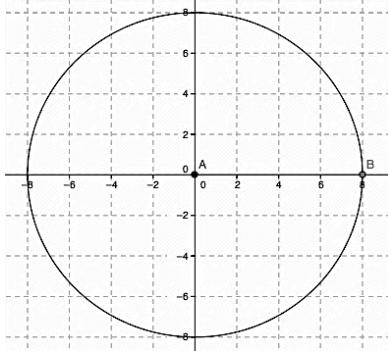
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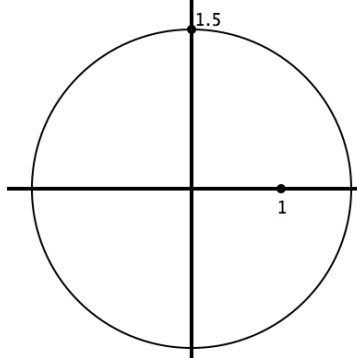
15.
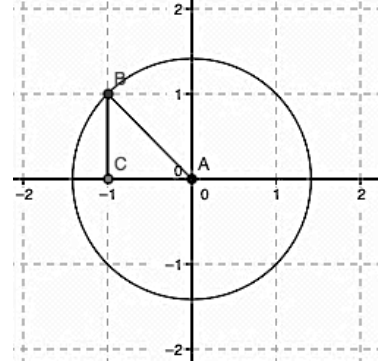
GO

Topic: Verifying Pythagorean triples

Identify which sets of numbers could be the sides of a right triangle. Show your work.

16. $\left\{ 9,12,15 \right\}$

17. $\left\{ 9,10,\sqrt{19} \right\}$

18. $\left\{ 1,\sqrt{3},2 \right\}$

19. $\left\{ 2,4,6 \right\}$

20. $\left\{ \sqrt{3},4,5 \right\}$

21. $\left\{ 10,24,26 \right\}$

22. $\left\{ \sqrt{2},\sqrt{7},3 \right\}$

23. $\left\{ 2\sqrt{2},5\sqrt{3},9 \right\}$

24. $\left\{ 4ab^{3}\sqrt{10},6ab^{3},14ab^{3} \right\}$

4

8.2 Getting Centered

A Solidify Understanding Task

Malik's family has decided to put in a new sprinkling system in their yard. Malik has volunteered to lay the system out. Sprinklers are available at the hardware store in the following sizes:

- Full circle, maximum 15' radius

- Half circle, maximum 15' radius

- Quarter circle, maximum 15' radius

All of the sprinklers can be adjusted so that they spray a smaller radius. Malik needs to be sure that the entire yard gets watered, which he knows will require that some of the circular water patterns will overlap. He gets out a piece of graph paper and begins with a scale diagram of the yard. In this diagram, the length of the side of each square represents 5 feet.

lawn

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5

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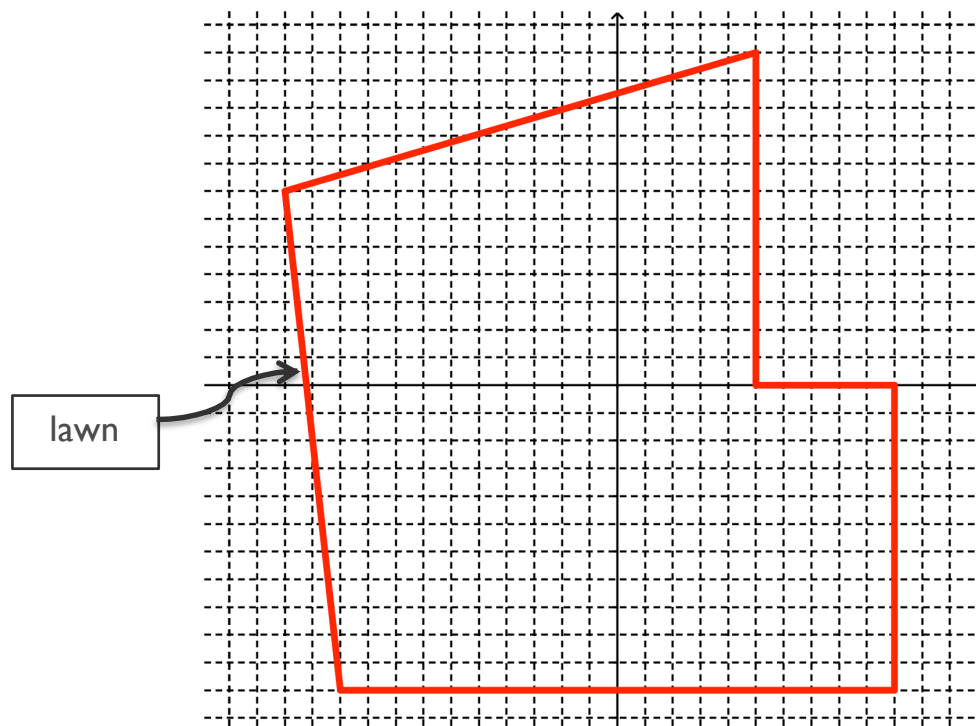
1. As he begins to think about locating sprinklers on the lawn, his parents tell him to try to cover the whole lawn with the fewest number of sprinklers possible so that they can save some money. The equation of the first circle that Malik draws to represent the area watered by the sprinkler is:

$$(x + 25)^2 + (y + 20)^2 = 225$$

   Draw this circle on the diagram using a compass.

2. Lay out a possible configuration for the sprinkling system that includes the first sprinkler pattern that you drew in #1.

3. Find the equation of each of the full circles that you have drawn.

Malik wrote the equation of one of the circles and just because he likes messing with the algebra, he did this:

   Original equation: $(x - 3)^2 + (y + 2)^2 = 225$

   $x^2 - 6x + 9 + y^2 + 4y + 4 = 225$

   $x^2 + y^2 - 6x + 4y - 212 = 0$

Malik thought, "That's pretty cool. It's like a different form of the equation. I guess that there could be different forms of the equation of a circle like there are different forms of the equation of a parabola or the equation of a line." He showed his equation to his sister, Sapana, and she thought he was nuts.   Sapana, said, "That's a crazy equation. I can't even tell where

the center is or the length of the radius anymore." Malik said, "Now it's like a puzzle for you. I'll give you an equation in the new form. I'll bet you can't figure out where the center is."

Sapana said, "Of course, I can. I'll just do the same thing you did, but work backwards."

4. Malik gave Sapana this equation of a circle:
$$x^2 + y^2 - 4x + 10y + 20 = 0$$

Help Sapana find the center and the length of the radius of the circle.

5. Sapana said, "Ok. I made one for you. What's the center and length of the radius for this circle?"

$$x^2 + y^2 + 6x - 14y - 42 = 0$$

6. Sapana said, "I still don't know why this form of the equation might be useful. When we had different forms for other equations like lines and parabolas, each of the various forms highlighted different features of the relationship." Why might this form of the equation of a circle be useful?

$$x^2 + y^2 + Ax + By + C = 0$$

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7

READY

Topic: Making perfect square trinomials

Fill in the number that completes the square. Then write the trinomial in factored form.

1. $x^2 + 6x +$ _____

2. $x^2 - 14x +$ _____

3. $x^2 - 50x +$ _____

4. $x^2 - 28x +$ _____

On the next set, leave the number that completes the square as a fraction. Then write the trinomial in factored form.

5. $x^2 - 11x +$ _____

6. $x^2 + 7x +$ _____

7. $x^2 + 15x +$ _____

8. $x^2 + \dfrac{2}{3}x +$ _____

9. $x^2 - \dfrac{1}{5}x +$ _____
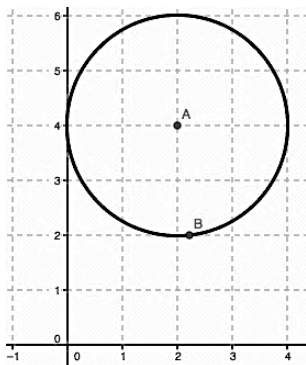
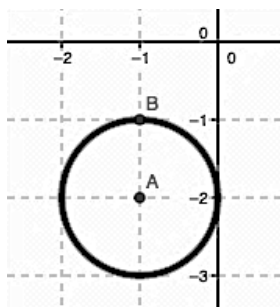10. $x^2 - \dfrac{3}{4}x +$ _____

SET

Topic: Writing equations of circles with center (h, k) and radius *r*.

Write the equation of each circle.

11.
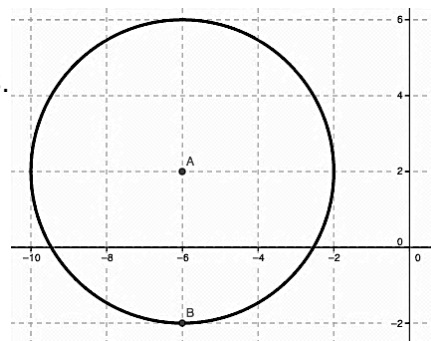

12.

13.

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Write the equation of the circle with the given center and radius. Then write it in expanded form.

14. Center: (5, 2) Radius: 13

15. Center: (-6, -10) Radius: 9

16. Center: (0, 8) Radius: 15

17. Center: (19, -13) Radius: 1

18. Center: (-1, 2) Radius: 10

19. Center: (-3, -4) Radius: 8

Go

Topic: Verifying if a point is a solution

Identify which point is a solution to the given equation. Show your work.

20. $y = \dfrac{4}{5}x - 2$ a. (-15, -14)   b. (10, 10)

21. $y = 3|x|$ a. (-4, -12)   b. $\left(-\sqrt{5}, 3\sqrt{5}\right)$

22. $y = x^2 + 8$ a. $\left(\sqrt{7}, 15\right)$ b. $\left(\sqrt{7}, -1\right)$

23. $y = -4x^2 + 120$ a. $\left(5\sqrt{3}, -180\right)$ b. $\left(5\sqrt{3}, 40\right)$

24. $x^2 + y^2 = 9$ a. $(8, -1)$ b. $\left(-2, \sqrt{5}\right)$

25. $4x^2 - y^2 = 16$ a. $\left(-3, \sqrt{10}\right)$ b. $\left(-2\sqrt{2}, 4\right)$

8.3 Circle Challenges

A Practice Understanding Task

Once Malik and Sapana started challenging each other with circle equations, they got a little more creative with their ideas. See if you can work out the challenges that they gave each other to solve. Be sure to justify all of your answers.

1. Malik's challenge:

    What is the equation of the circle with center (-13,-16) and containing the point (-10,-16) on the circle?

2. Sapana's challenge:

    The points (0, 5) and (0,-5) are the endpoints of the diameter of a circle. The point (3, y) is on the circle. What is a value for y?

3. Malik's challenge:

    Find the equation of a circle with center in the first quadrant and is tangent to the lines $x = 8$, $y = 3$, and $x = 14$.

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4. Sapana's challenge:

   The points (4,-1) and (-6,7) are the endpoints of the diameter of a circle. What is the equation of the circle?

5. Malik's challenge:

   Is the point (5,1) inside, outside, or on the circle $x^2 - 6x + y^2 + 8y = 24$? How do you know?

6. Sapana's challenge:

   The circle defined by $(x - 1)^2 + (y + 4)^2 = 16$ is translated 5 units to the left and 2 units down. Write the equation of the resulting circle.

7. Malik's challenge:

There are two circles, the first with center (3,3) and radius r_1, and the second with center (3, 1) and radius r_2.

a. Find values r_1 and r_2 so that the first circle is completely enclosed by the second circle.

b. Find one value of r_1 and one value of r_2 so that the two circles intersect at two points.

c. Find one value of r_1 and one value of r_2 so that the two circles intersect at exactly one point.

READY

Topic: Finding the distance between two points

Simplify. Use the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ **to find the distance between the given points. Leave your answer in simplest radical form.**

1. $A(18,-12)$ $B(10,4)$ 2. $G(-11,-9)$ $H(-3,7)$ 3. $J(14,-20)$ $K(5,5)$

4. $M(1,3)$ $P(-2,7)$ 5. $Q(8,2)$ $R(3,7)$

6. $S(-11,2\sqrt{2})$ $T(-5,-4\sqrt{2})$ 7. $W(-12,-2\sqrt{2})$ $Z(-7,-3\sqrt{2})$

SET

Topic: Writing equations of circles

Use the information provided to write the equation of the circle in standard form,
$$(x - h)^2 + (y - h)^2 = r^2$$

8. Center $(-16,-5)$ and the circumference is 22π

9. Center $(13,-27)$ and the area is 196π

10. Diameter measures 15 units and the center is at the intersection of y = x + 7 and y = 2x – 5

11. Lies in quadrant 2 Tangent to $x = -12 \; and \; x = -4$

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12. Center (-14 , 9) Point on circle (1 , 11)

13. Center lies on the y axis   Tangent to y = -2 and y = -17

14. Three points on the circle are $\left(-8,5\right),\left(3,-6\right),\left(14,5\right)$

15. I know three points on the circle are (-7,6), (9,6), and (-4,13). I think that the equation of the circle is $\left(x-1\right)^2 +\left(y-6\right)^2 = 64$. Is this the correct equation for the circle? Justify your answer.

GO

Topic: Finding the value of **B** in a quadratic in the form of $Ax^2 + Bx + C$ in order to create a perfect square trinomial.

Find the value of *B* that will make a perfect square trinomial. Then write the trinomial in factored form.

16. $x^2 +$ _____ $x + 36$

17. $x^2 +$ _____ $x + 100$

18. $x^2 +$ _____ $x + 225$

19. $9x^2 +$ _____ $x + 225$

20. $16x^2 +$ _____ $x + 169$

21. $x^2 +$ _____ $x + 5$

22. $x^2 +$ _____ $x + \dfrac{25}{4}$

23. $x^2 +$ _____ $x + \dfrac{9}{4}$

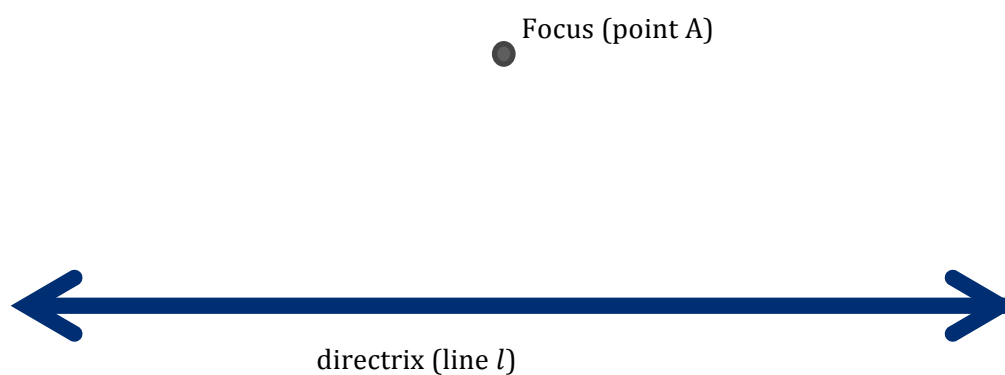24. $x^2 +$ _____ $x + \dfrac{49}{4}$

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8.4 Directing Our Focus

A Develop

Understanding Task

On a board in your classroom, your teacher has set up a point and a line like this:

Focus (point A)

directrix (line *l*)

We're going to call the line a directrix and the point a focus. They've been labeled on the drawing.

Similar to the circles task, the class is going to construct a geometric figure using the focus (point A) and directrix (line *l*).

1. Cut two pieces of string with the same length.

2. Mark the midpoint of each piece of string with a marker.

3. Position the string on the board so that the midpoint is equidistant from the focus (point A) and the directrix (line l), which means that it must be perpendicular to the directrix. While holding the string in this position, put a pin through the midpoint. Depending on the size of your string, it will look something like this:
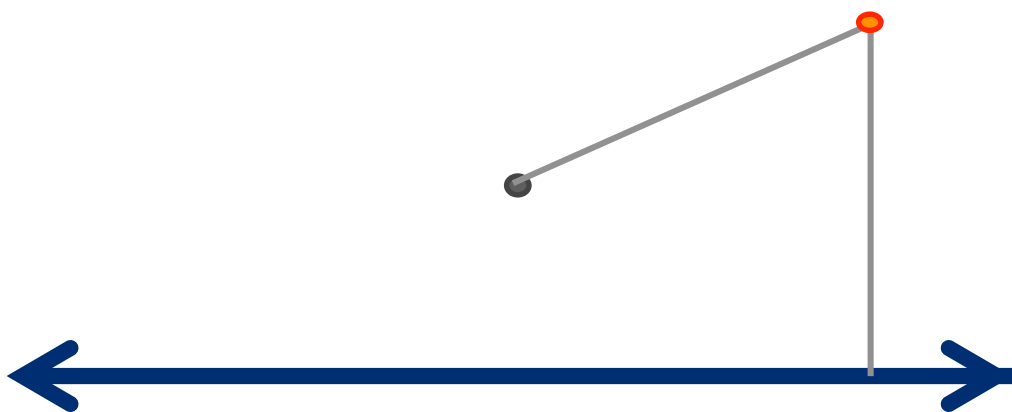
4. Using your second string, use the same procedure to post a pin on the other side of the focus.

5. As your classmates post their strings, what geometric figure do you predict will be made by the tacks (the collection of all points like (x, y) show in the figure above)? Why?

6. Where is the vertex of the figure located? How do you know?

7. Where is the line of symmetry located? How do you know?

8. Consider the following construction with focus point A and the x-axis as the directrix. Use a ruler to complete the construction of the parabola in the same way that the class constructed the parabola with string.
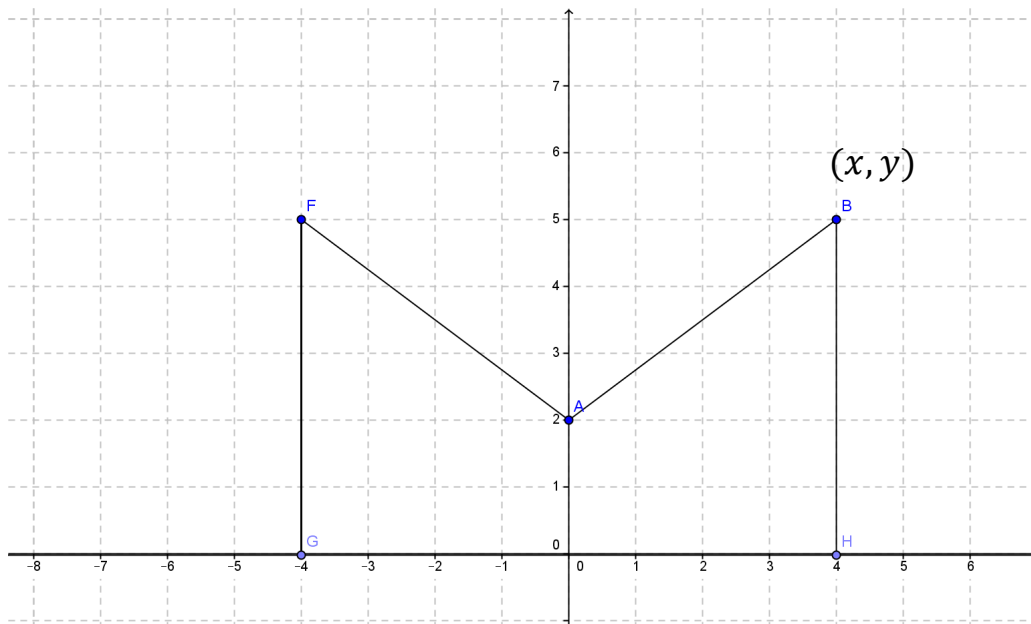
9. You have just constructed a parabola based upon the definition: A parabola is the set of all points (x, y) equidistant from a line l (the directrix) and a point not on the line (the focus). Use this definition to write the equation of the parabola above, using the point (x, y) to represent any point on the parabola.

10. How would the parabola change if the focus was moved up, away from the directrix?

11. How would the parabola change if the focus were to be moved down, toward the directrix?

12. How would the parabola change if the focus were to be moved down, below the directrix?
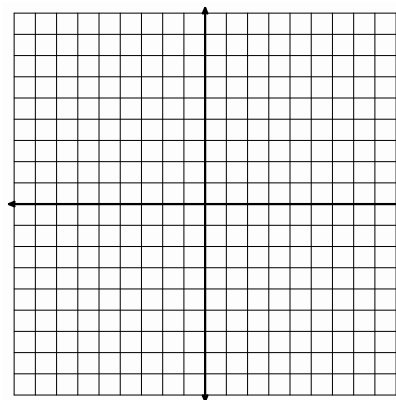
READY

Topic: Graphing Quadratics

Graph each set of functions on the same coordinate axes. Describe in what way the graphs are the same and in what way they are different.
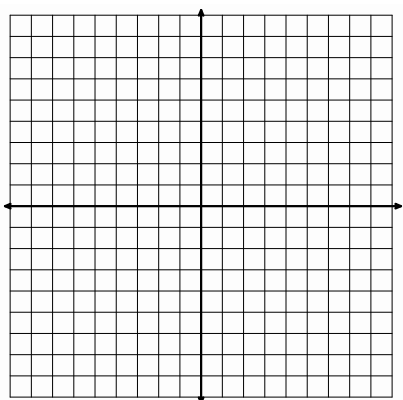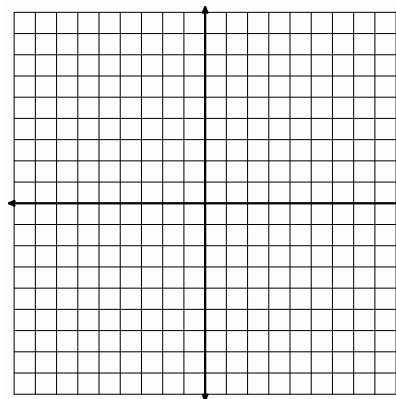
1. $y = x^2$, $y = 2x^2$, $y = 4x^2$
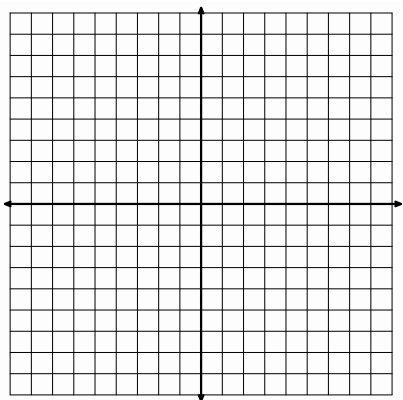
2. $y = \dfrac{1}{4}x^2$, $y = -x^2$, $y = -4x^2$

3. $y = \dfrac{1}{4}x^2$, $y = x^2 - 2$, $y = \dfrac{1}{4}x^2 - 2$, $y = 4x^2 - 2$

4. $y = x^2$, $y = -x^2$, $y = x^2 + 2$, $y = -x^2 + 2$

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Topic: Sketching a parabola using the conic definition.

Use the conic definition of a parabola to sketch a parabola defined by the given focus *F* and the equation of the directrix.

Begin by graphing the focus, the directrix, and point P₁. Use the distance formula to find FP₁ and find the vertical distance between P₁ and the directrix by identifying point H on the directrix and counting the distance. Locate the point P₂, (the point on the parabola that is a reflection of P₁ across the axis of symmetry.) Locate the vertex V. Since the vertex is a point on the parabola, it must lie equidistant between the focus and the directrix. Sketch the parabola. Hint: the parabola always "hugs" the focus.
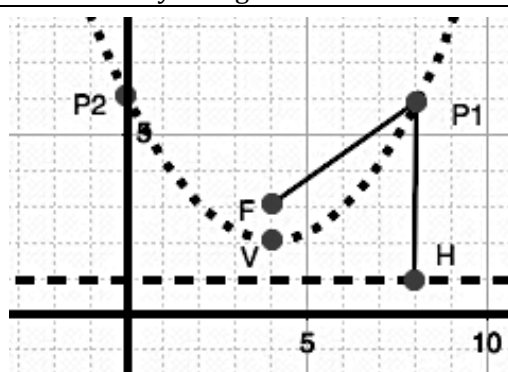
Example: *F* (4,3), *P₁* (8, 6), y = 1

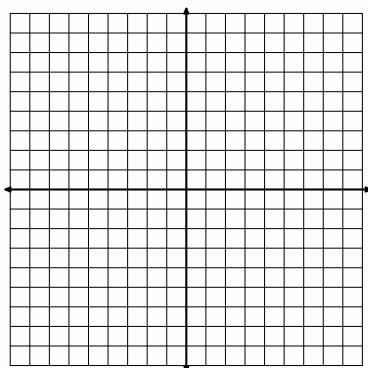$$FP_1 = \sqrt{(4-8)^2 + (3-6)^2} = \sqrt{16+9} = \sqrt{25} = 5$$
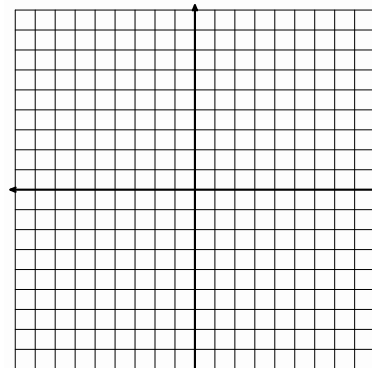
P₁H = 5

P₂ is located at (0, 6)

V is located at (4, 2)

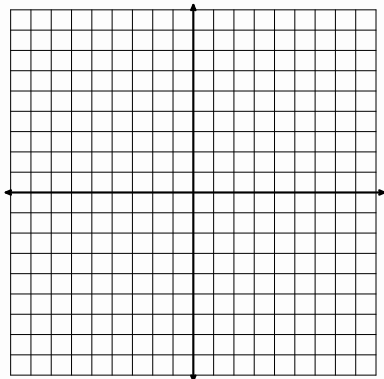5. *F* (1,-1), *P₁* (3, -1)   y = -3

6. *F* (-5,3), *P₁* (-1, 3) y = 7


7. *F* (2,1), *P₁* (-2, 1) y =-3

8. *F* (1, -1), *P₁* (-9, -1) y = 9
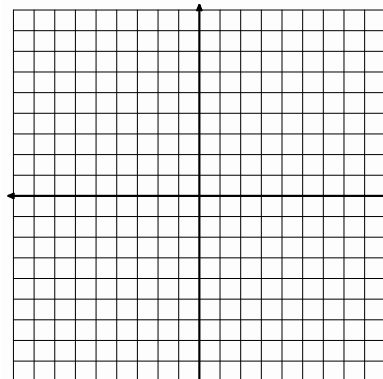
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9. Find a square piece of paper (a post-it note will work).
Fold the square in half vertically and put a dot anywhere
on the fold. Let the edge of the paper be the directrix and
the dot be the focus. Fold the edge of the paper (the directrix) up
to the dot repeatedly from different points along the edge.
The fold lines between the focus and the edge should make
a parabola.
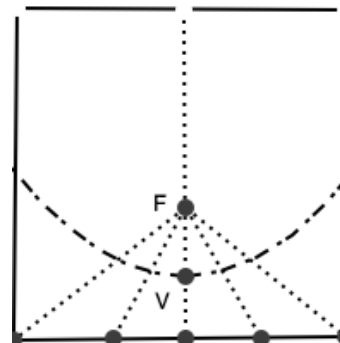
Experiment with a new paper and move the focus.
Use your experiments to answer the following questions.

10. How would the parabola change if the focus were moved up, away from the directrix?

11. How would the parabola change if the focus were moved down, toward the directrix?

12. How would the parabola change if the focus were moved down, below the directrix?

GO

Topic: Finding the center and radius of a circle.
Write each equation so that it shows the center (h, k) and radius r of the circle. This called the
standard form of a circle. $(x - h)^2 + (y - k)^2 = r^2$

13. $x^2 + y^2 + 4y - 12 = 0$ 14. $x^2 + y^2 - 6x - 3 = 0$

15. $x^2 + y^2 + 8x + 4y - 5 = 0$ 16. $x^2 + y^2 - 6x - 10y - 2 = 0$

17. $x^2 + y^2 - 6y - 7 = 0$ 18. $x^2 + y^2 - 4x + 8y + 6 = 0$

19. $x^2 + y^2 - 4x + 6y - 72 = 0$ 20. $x^2 + y^2 + 12x + 6y - 59 = 0$

21. $x^2 + y^2 - 2x + 10y + 21 = 0$ 22. $4x^2 + 4y^2 + 4x - 4y - 1 = 0$

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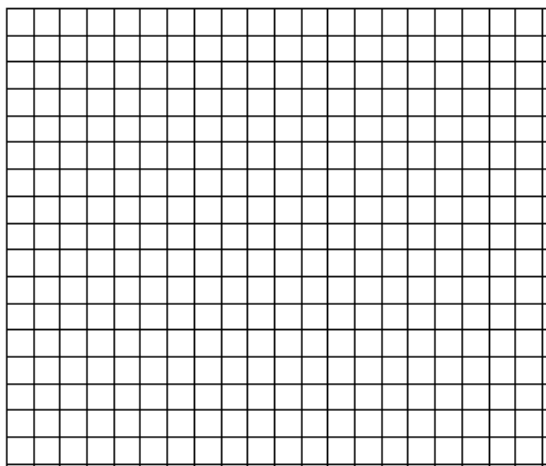
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8.5 Functioning With Parabolas

A *Solidify Understanding Task*

Sketch the graph of each parabola(accurately), find the vertex and use the geometric definition of a parabola to find the equation of these parabolas.
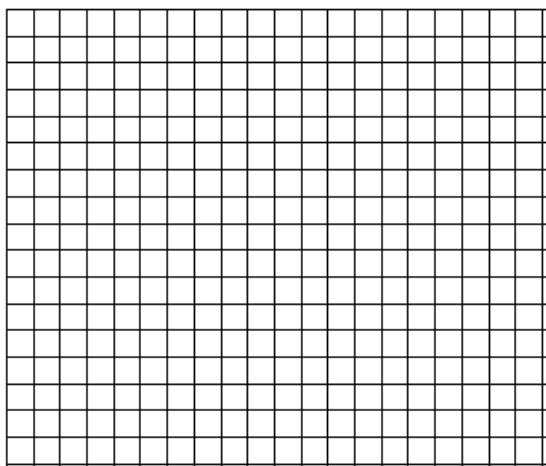
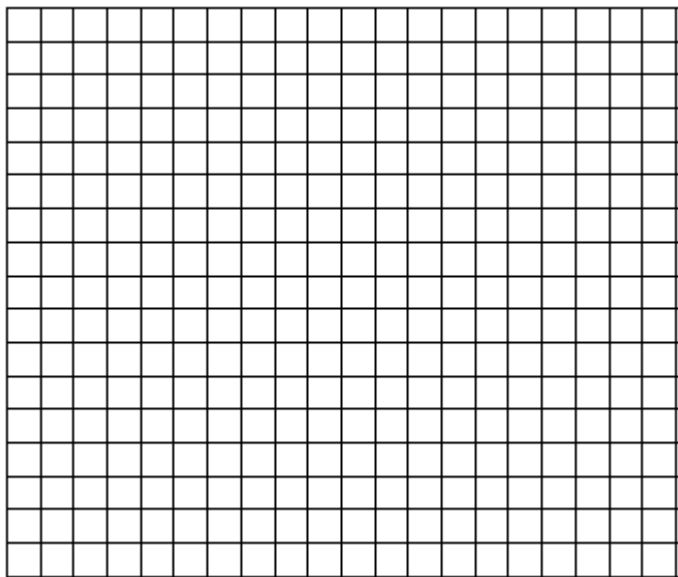1. Directrix $y = -4$, Focus A(2, -2)

Vertex _____

Equation:

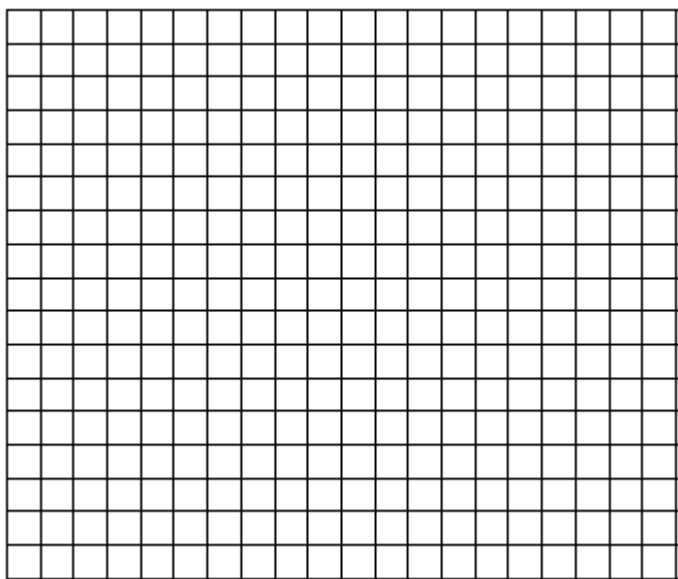2. Directrix $y = 2$, Focus A(-1, 0)

Vertex _____

Equation:

3. Directrix $y = 3$, Focus A(1, 7)

Vertex _____

Equation:

4. Directrix $y = 3$, Focus A(2, -1)

Vertex _____

Equation:

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5. Given the focus and directrix, how can you find the vertex of the parabola?

6. Given the focus and directrix, how can you tell if the parabola opens up or down?

7. How do you see the distance between the focus and the vertex (or the vertex and the directrix) showing up in the equations that you have written?

8. Describe a pattern for writing the equation of a parabola given the focus and directrix.

9. Annika wonders why we are suddenly thinking about parabolas in a completely different way than when we did quadratic functions. She wonders how these different ways of thinking match up. For instance, when we talked about quadratic functions earlier we started with $y = x^2$. "Hmmmm. …. I wonder where the focus and directrix would be on this function," she thought. Help Annika find the focus and directrix for $y = x^2$.

10. Annika thinks, "Ok, I can see that you can find the focus and directrix for a quadratic function, but what about these new parabolas. Are they quadratic functions? When we work with families of functions, they are defined by their rates of change. For instance, we can tell a linear function because it has a constant rate of change." How would you answer Annika? Are these new parabolas quadratic functions? Justify your answer using several representations and the parabolas in problems 1-4 as examples.

READY

Topic: Standard form of a quadratic.

Verify that the given point lies on the graph of the parabola described by the equation. (Show your work.)

1. $(6,0)$ $y = 2x^2 - 9x - 18$

2. $(-2,49)$ $y = 25x^2 + 30x + 9$

3. $(5,53)$ $y = 3x^2 - 4x - 2$

4. $(8,2)$ $y = \dfrac{1}{4}x^2 - x - 6$

SET

Topic: Equation of parabola based on the geometric definition

5. Verify that $(y-1) = \dfrac{1}{4}x^2$ is the equation

of the parabola in *Figure 1* by plugging in the 3 points V (0,1), C (4,5) and E (2,2).
Show your work for each point!
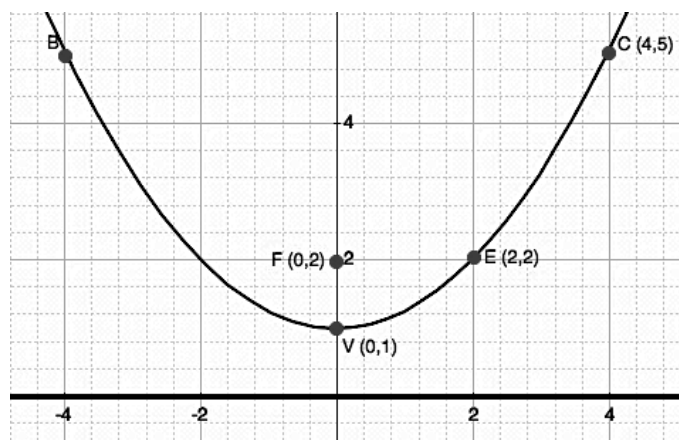
Figure 1

6. If you didn't know that (0,1) was the vertex of the parabola, could you have found it by just looking at the equation? Explain.

7. Use the diagram in *Figure 2* to derive the general equation of a parabola based on the **geometric definition** of a parabola. Remember that the definition states that MF = MQ.
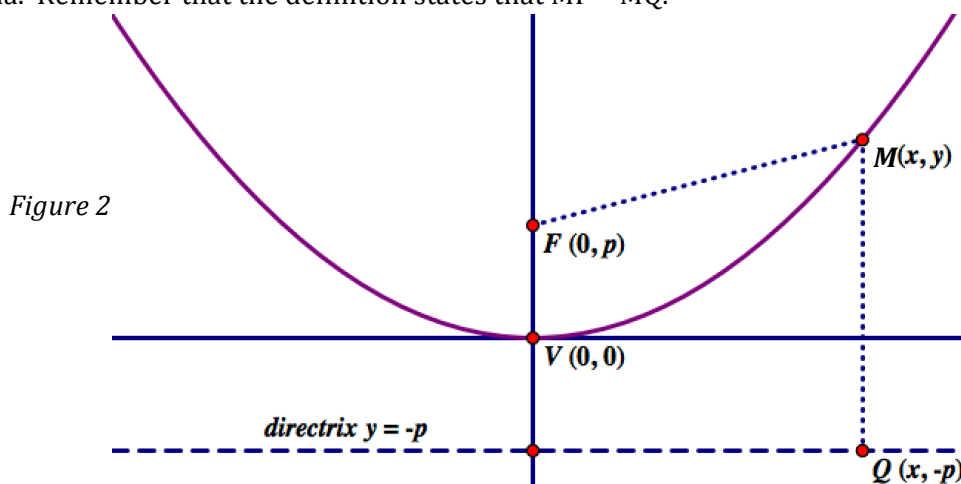
Figure 2

8. Recall the equation in #5, $(y-1) = \dfrac{1}{4}x^2$, what is the value of **p**?

9. In general, what is the value of **p** for any parabola?

10. In *Figure 3*, the point M is the same height as the focus and $\overline{FM} \cong \overline{MR}$. How do the coordinates of this point compare with the coordinates of the focus?
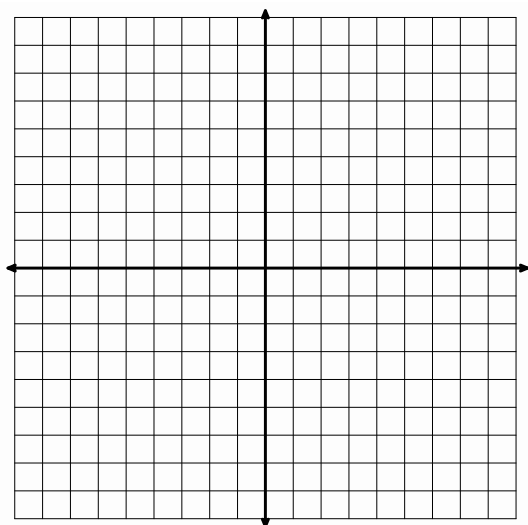Fill in the missing coordinates for M and R in the diagram.

Figure 3
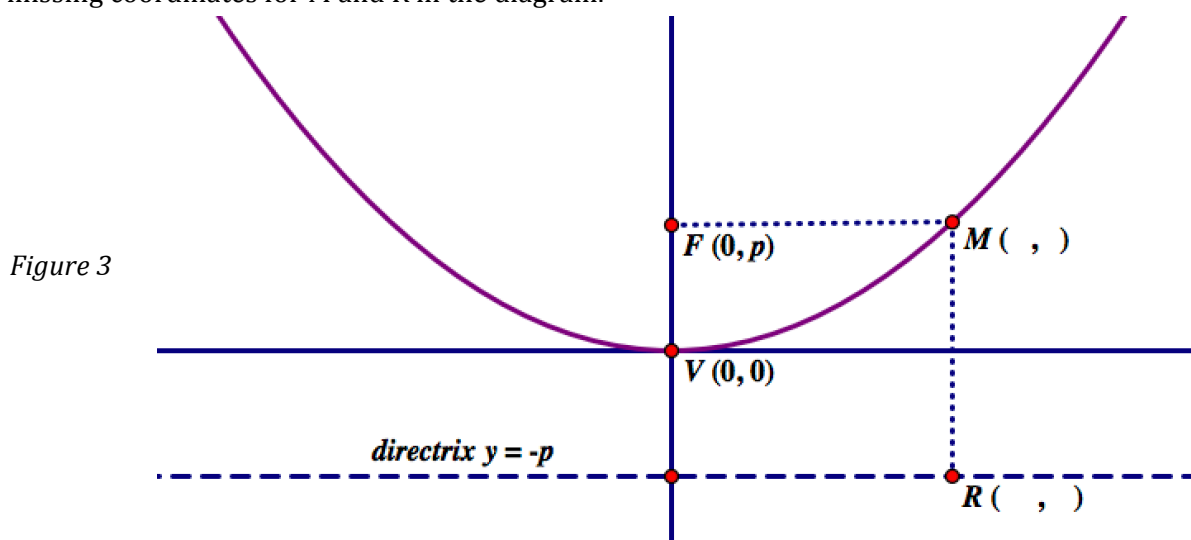
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Sketch the graph by finding the vertex and the point M and R (the reflection of M) as defined in the diagram above. Use the geometric definition of a parabola to find the equation of these parabolas.
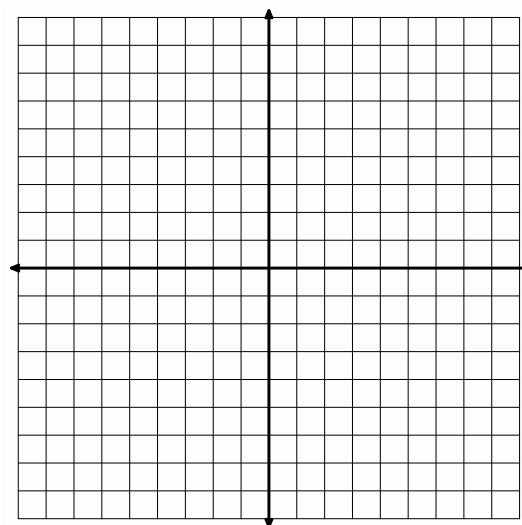
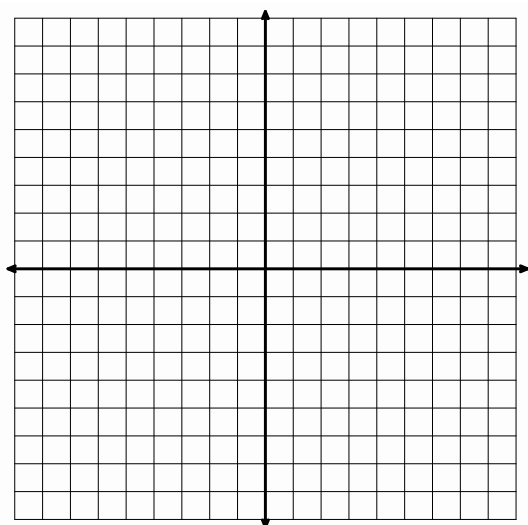11. Directrix y = 9, Focus F(-3, 7)

Vertex _____
Equation _____

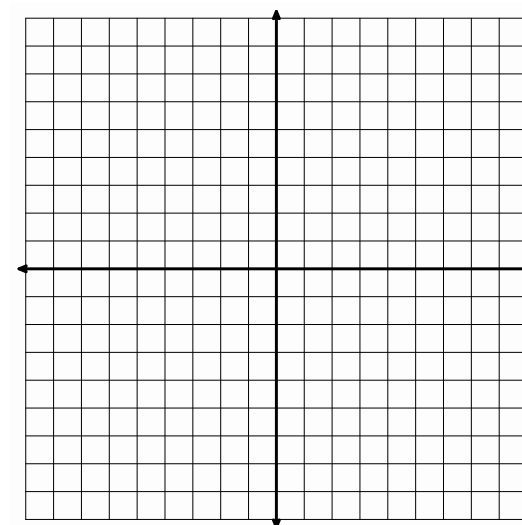12. Directrix y = -6, Focus F(2, -2)

Vertex _____
Equation _____

13. Directrix y = 5, Focus F(-4, -1)

Vertex _____
Equation _____

14. Directrix y = -1, Focus F(4, -3)

Vertex _____
Equation _____

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Topic: Finding minimum and minimum values for quadratics

Find the maximum or minimum value of the quadratic. Indicate which it is.

15. $y = x^2 + 6x - 5$

16. $y = 3x^2 - 12x + 8$

17. $y = -\dfrac{1}{2}x^2 + 10x + 13$

18. $y = -5x^2 + 20x - 11$

19. $y = \dfrac{7}{2}x^2 - 21x - 3$

20. $y = -\dfrac{3}{2}x^2 + 9x + 25$

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8.6 Turn It Around

A Solidify Understanding Task

Annika is thinking more about the geometric view of parabolas that she has been working on in math class. She thinks, "Now I see how all the parabolas that come from graphing quadratic functions could also come from a given focus and directrix. I notice that all the parabolas have opened up or down when the directrix is horizontal. I wonder what would happen if I rotated the focus and directrix 90 degrees so that the directrix is vertical. How would that look? What would the equation be? Hmmm…." So Annika starts trying to construct a parabola with a vertical directrix. Here's the beginning of her drawing. Use a ruler to complete Annika's drawing.
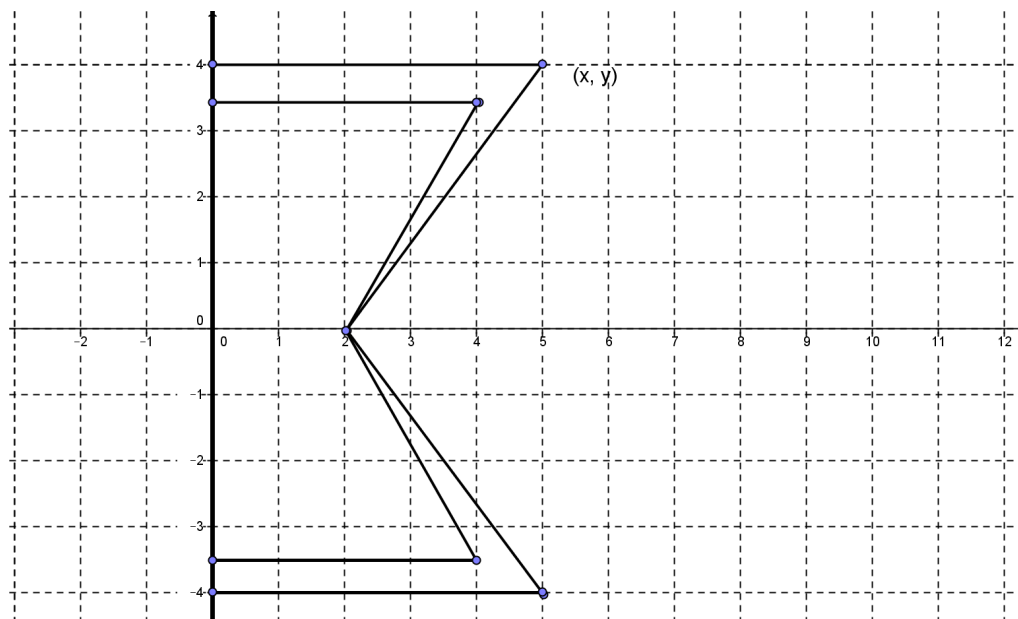
1. Use the definition of a parabola to write the equation of Annika's parabola.

2. What similarities do you see to the equations of parabolas that open up or down? What differences do you see?

3. Try another one: Write the equation of the parabola with directrix $x = 4$ and focus $(0, 3)$.

4. One more for good measure: Write the equation of the parabola with directrix $x = -3$ and focus $(-2, -5)$.

5. How can you predict if a parabola will open left, right, up, or down?

6. How can you tell how wide or narrow a parabola is?

7. Annika has two big questions left. Write and explain your answers to these questions.
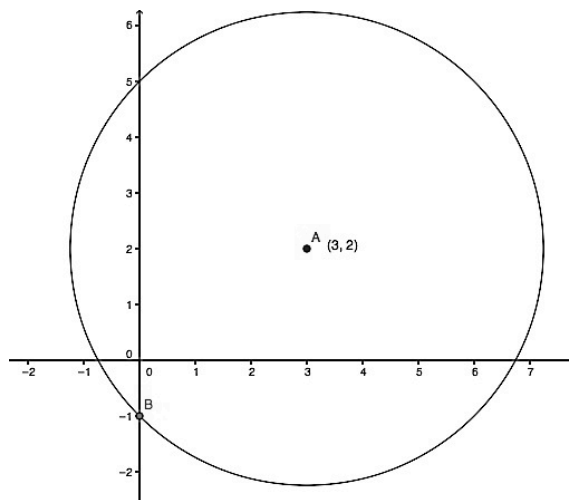   a. Are all parabolas functions?

    b. Are all parabolas similar?

READY

Topic: Review of Circles

Use the given information to write the equation of the circle in standard form.

1. Center: (-5, -8) , Radius: 11

2. Endpoints of the diameter: (6, 0) and (2, -4)

3. Center (-5, 4) : Point on the circle (-9, 1)

4. Equation of the circle in the diagram to the right.

SET

Topic: Writing equations of horizontal parabolas

Use the focus F, point M, a point on the parabola, and the equation of the directrix to sketch the parabola (label your points) and write the equation. Put your equation in the form

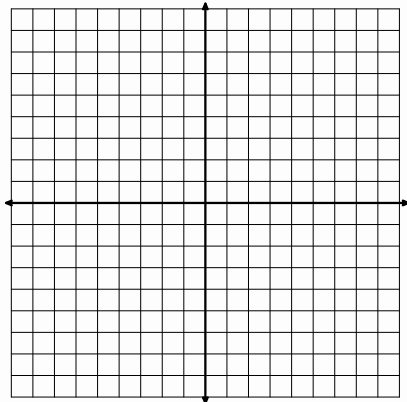$$x = \frac{1}{4p}(y - k)^2 + h \text{ where "p" is the distance from the focus to the vertex.}$$

5. **F** (1,0), **M** (1,4) x = -3

6. **F** (3,1), **M** (3,-5) x = 9

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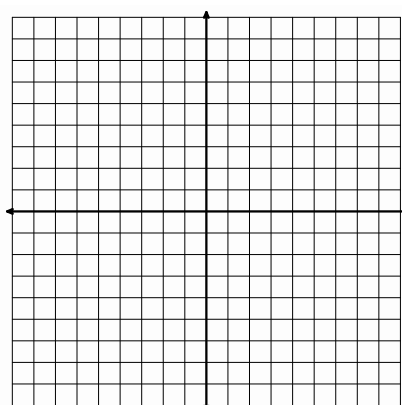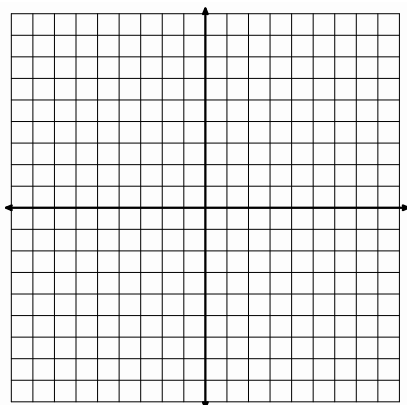
7. **F** (7,-5), **M** (4,-1) x = 9

8. **F** (-1,2), **M** (6,-9) x = -7
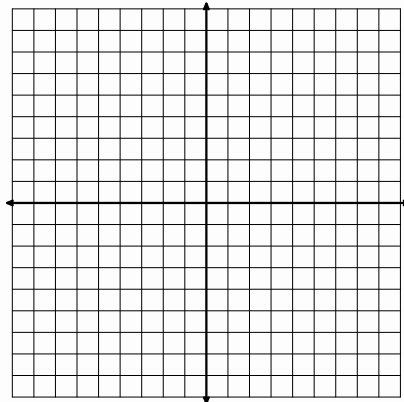
GO

Topic: Identifying key features of a quadratic written in vertex form

State (a) the coordinates of the vertex, (b) the equation of the axis of symmetry, (c) the domain, and (d) the range for each of the following functions.

9. $f(x)=(x-3)^2+5$

10. $f(x)=(x+1)^2-2$

11. $f(x)=-(x-3)^2-7$

12. $f(x)=-3\left(x-\dfrac{3}{4}\right)^2+\dfrac{4}{5}$

13. $f(x)=\dfrac{1}{2}(x-4)^2+1$

14. $f(x)=\dfrac{1}{4}(x+2)^2-4$

15. Compare the vertex form of a quadratic to the geometric definition of a parabola based on the focus and directrix. Describe how they are similar and how they are different.

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