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Transforming Mathematics Education

# SECONDARY MATH THREE

*An Integrated Approach*

MODULE 7 HONORS

# Trigonometric Functions, Equations & Identities

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**The Mathematics Vision Project**

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# 7.1 High Noon and Sunset Shadows

### *A Develop Understanding Task*



In this task we revisit the amusement park Ferris wheel that caused Carlos so much anxiety. Recall the following facts from previous tasks:

- The Ferris wheel has a radius of 25 feet
- The center of the Ferris wheel is 30 feet above the ground
- The Ferris wheel makes one complete rotation counterclockwise every 20 seconds

The amusement park Ferris wheel is located next to a high-rise office complex. At sunset, the moving carts cast a shadow on the exterior wall of the high-rise building. As the Ferris wheel turns, you can watch the shadow of a rider rise and fall along the surface of the building. In fact, you know an equation that would describe the motion of this "sunset shadow."

1. Write the equation of this "sunset shadow."

At noon, when the sun is directly overhead, a rider casts a shadow that moves left and right along the ground as the Ferris wheel turns. In fact, you know an equation that would describe the motion of this "high noon shadow."

2. Write the equation of this "high noon shadow."

3. Based on your previous work, you probably wrote these equations in terms of the angle of rotation being measured in degrees. Revise you equations so the angle of rotation is measured in radians.

    a. The “sunset shadow” equation in terms of radians:

    b. The “high noon shadow” equation in terms of radians:

4. In the equations you wrote in question 3, where on the Ferris wheel was the rider located at time $t = 0$? (We will refer to the position as the rider’s *initial position* on the wheel.)

5. Revise your equations from question 3 so that the rider’s initial position at $t = 0$ is at the top of the wheel.

    a. The “sunset shadow” equation, initial position at the top of the wheel:

    b. The “high noon shadow” equation, initial position at the top of the wheel:

6. Revise your equations from question 3 so that the rider’s initial position at $t = 0$ is at the bottom of the wheel.

    a. The “sunset shadow” equation, initial position at the bottom of the wheel:

    b. The “high noon shadow” equation, initial position at the bottom of the wheel:

7. Revise your equations from question 3 so that the rider's initial position at $t = 0$ is at the point farthest to the left of the wheel.

    a. The "sunset shadow" equation, initial position at the point farthest to the left of the wheel:

    b. The "high noon shadow" equation, initial position at the point farthest to the left of the wheel:

8. Revise your equations from question 3 so that the rider's initial position at $t = 0$ is halfway between the farthest point to the right on the wheel and the top of the wheel.

    a. The "sunset shadow" equation, initial position halfway between the farthest point to the right on the wheel and the top of the wheel:

    b. The "high noon shadow" equation, initial position halfway between the farthest point to the right on the wheel and the top of the wheel:

9. Revise your equations from question 3 so that the wheel rotates twice as fast.

    a. The "sunset shadow" equation for the wheel rotating twice as fast:

    b. The "high noon shadow" equation for the wheel rotating twice as fast:

10. Revise your equations from question 3 so that the radius of the wheel is twice as large and the center of the wheel is twice as high.

    a. The "sunset shadow" equation for a radius twice as large and the center twice as high:

    b. The "high noon shadow" equation for a radius twice as large and the center twice as high:

11. Carlos wrote his "sunset equation" for the height of the rider in question #5 as $h(t) = 50\sin\left(\frac{\pi}{10}t + \frac{\pi}{2}\right) + 30$. Clarita wrote her equation for the same problem as $h(t) = 50\sin\left(\frac{\pi}{10}(t+5)\right) + 30$.

    a. Are both of these equations equivalent? How do you know?

    b. Carlos says his equation represents starting the rider at an initial position at the top of the wheel. What does Clarita's equation represent?

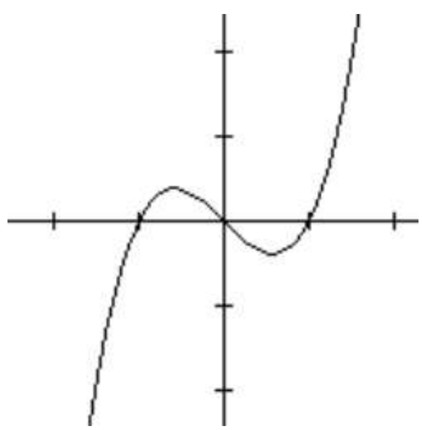
READY, SET, GO! Name Period Date

## READY

Topic: Recalling invertible functions and even and odd functions

**Indicate which of the following functions have an inverse that is a function. If the function has an inverse, sketch it in.** (Remember, the inverse will reflect across the y = x line. Sketch that in, too.) Finally, label each one as *even, odd*, or *neither*. Recall that an *even* function is symmetric with the y-axis, while an *odd* function is symmetric with respect to the origin.

1,



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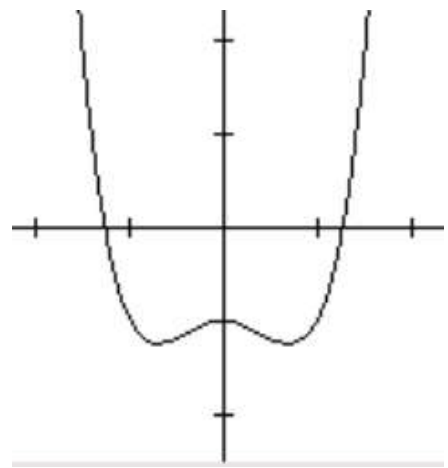

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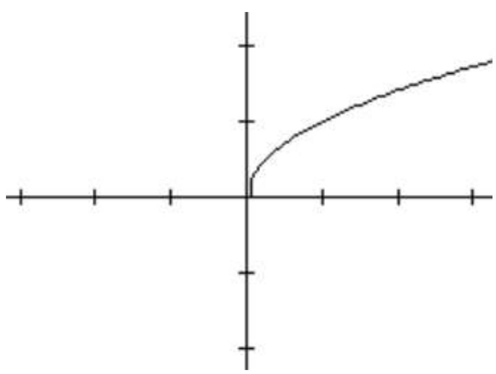

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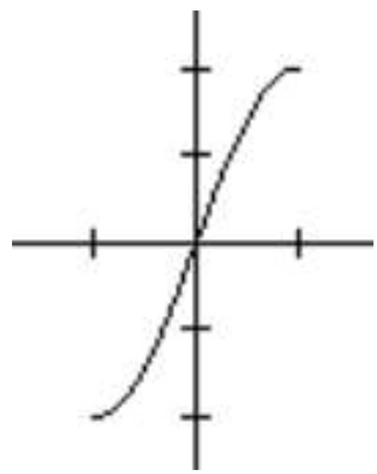

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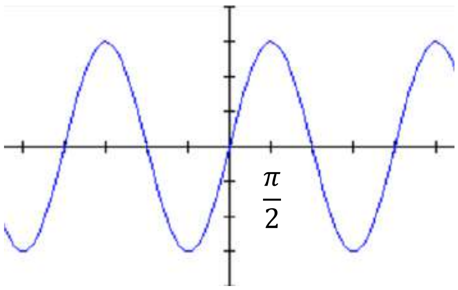
## SET

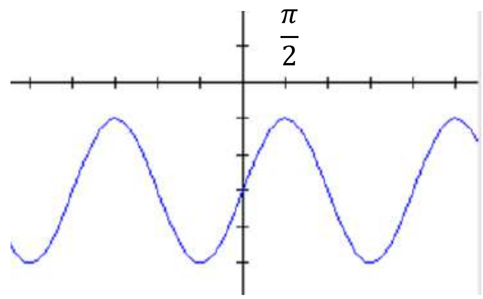
Topic: Connecting transformed trig graphs with their equations

**State the period, amplitude, vertical shift, and phase shift of the function shown in the graph. Then write the equation. Use the same trigonometric function as the one that is given.**

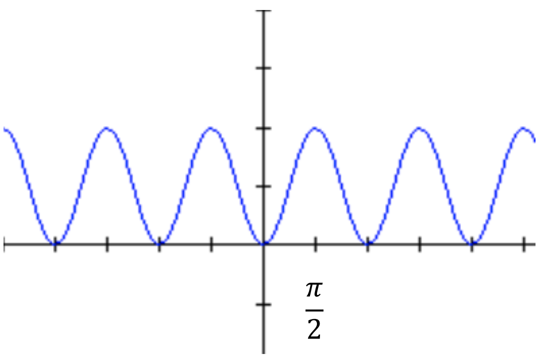
5. $y = sin\, x$



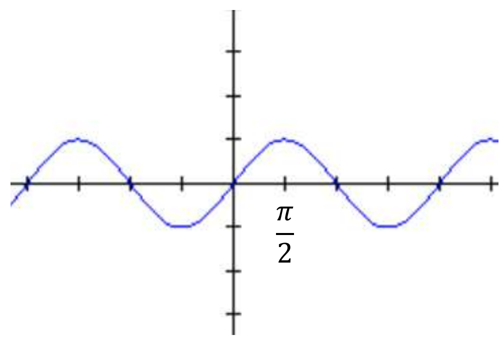

6. $y = sin\, x$



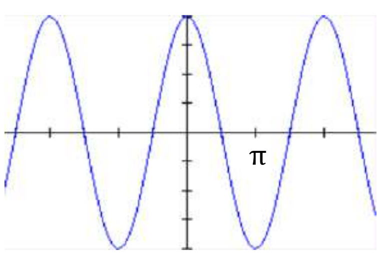

7. $y = cos\, x$




8. $y = cos\, x$




9. $y = sin\, x$



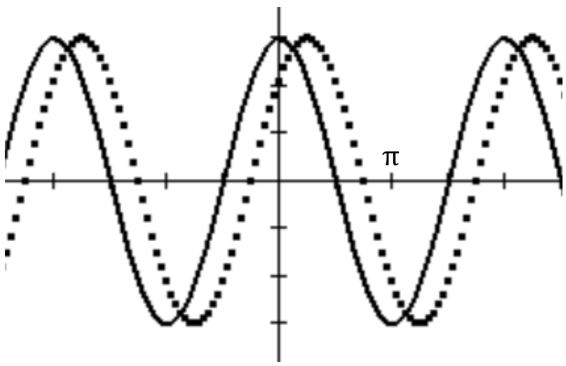

10. The cofunction identity states that $\sin\theta = \cos(90° - \theta)$ *and* $\sin(\theta - 90°) = \cos\theta$. How does this identity relate to the graph in #9?

Explain where you would see this identity in a right triangle.

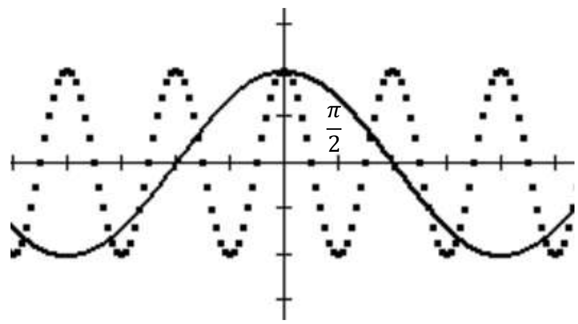
**Describe the relationships between the graphs of *f (x)* - solid and *g (x)* - dotted.**

**Then write their equations.**

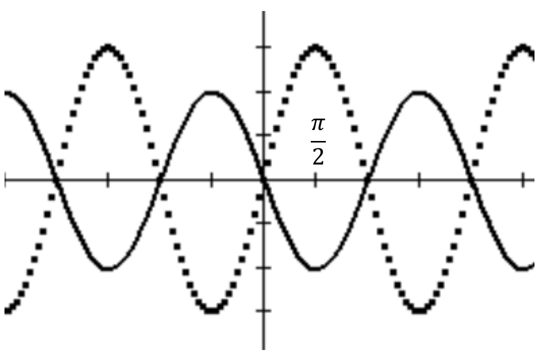
11.



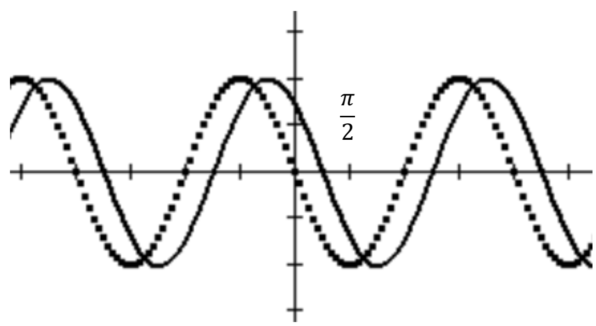

12.




13. This graph could be interpreted as a shift or a reflection. Write the equations both ways.

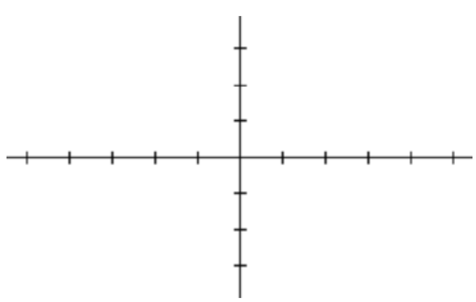



14.

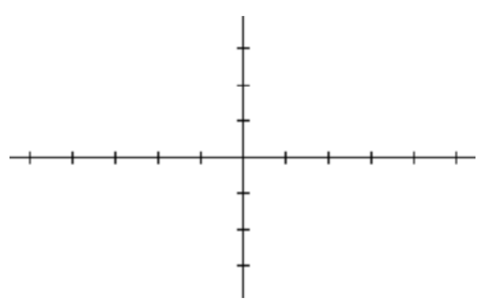



**Sketch the graph of the function.**
**(Include 2 full periods. Label the scale of your horizontal axis.)**

15. $y = 3\sin\left(x - \frac{\pi}{2}\right)$



16. $y = -2\cos(x + \pi)$



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## GO

Topic: Finding angles of rotation for the same trig ratio

**Name two values for $\theta$ (angles of rotation) that have the given trig ratio. $0 < \theta \leq 2\pi$.**

17. $\sin\theta = \frac{\sqrt{2}}{2}$

18. $\cos\theta = \frac{\sqrt{2}}{2}$

19. $\cos\theta = -\frac{1}{2}$

20. $\sin\theta = 0$

21. $\sin\theta = -\frac{\sqrt{3}}{2}$

22. $\cos\theta = -\frac{\sqrt{3}}{2}$

23. For which angles of rotation does $\sin\theta = \cos\theta$ ?

# 7.2 High Tide



## *A Solidify Understanding Task*

Perhaps you have built an elaborate sand castle at the beach only to have it get swept away by the in-coming tide.

Spring break is next week and you are planning another trip to the beach. This time you decide to pay attention to the tides so that you can keep track of how much time you have to build and admire your sand castle.

You have a friend who is in calculus who will be going on spring break with you. You give your friend some data from the almanac about high tides along the ocean, as well as a contour map of the beach you intend to visit, and ask her to come up with an equation for the water level on the beach on the day of your trip. According to your friend's analysis, the water level on the beach will fit this equation:

$$f(t) = 20\, sin\left(\frac{\pi}{6}t\right)$$

In this equation, $f(t)$ represents how far the waterline is above or below its average position. The distance is measured in feet, and $t$ represents the elapsed time (in hours) since midnight.

1. What is the highest up the beach (compared to its average position) that the waterline will be during the day? (This is called *high tide*.) What is the lowest that the waterline will be during the day? (This is called *low tide*.)

2. Suppose you plan to build your castle right on the average waterline just as the water has moved below that line. How much time will you have to build your castle before the incoming tide destroys your work?

3. Suppose you want to build your castle 10 feet below the average waterline to take advantage of the damp sand. What is the maximum amount of time you will have to make your castle? How can you convince your friend that your answer is correct?

4. Suppose you want to build your castle 15 feet above the average waterline to give you more time to admire your work. What is the maximum amount of time you will have to make your castle? How can you convince your friend that your answer is correct?

5. You may have answered the previous questions using a graph of the tide function. Is there a way you could use algebra and the inverse sine function to answer these questions. If so, show your work.

   a. Algebraic work for question 3:

b. Algebraic work for question 4:

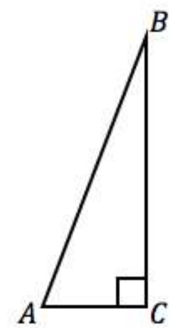
6. Suppose you decide you only need two hours to build and admire your castle. What is the lowest point on the beach where you can build it? How can you convince your friend that your answer is correct?

**READY, SET, GO!** Name Period Date

## READY

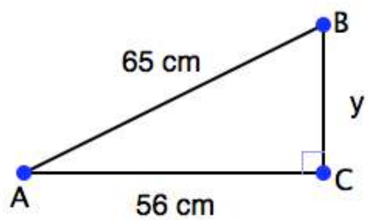
Topic: Calculating tangent in right angle trigonometry

Recall that the right triangle definition of the tangent ratio is:

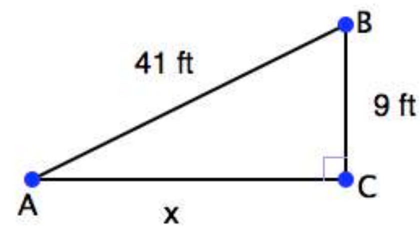



$$\tan A = \frac{\textit{length of side opposite angle } A}{\textit{length of side adjacent to angle } A}$$

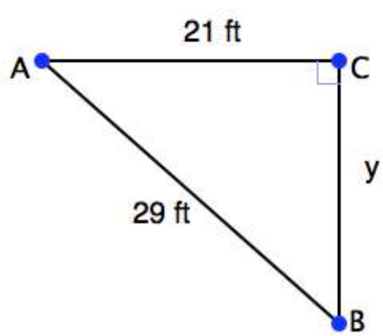
1. Find $\tan A$ *and* $\tan B$.



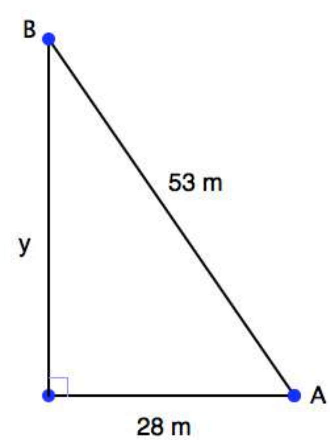

2. Find $\tan A$ *and* $\tan B$.




3. Find $\tan A$ *and* $\tan B$.




4. Find $\tan A$ *and* $\tan B$.

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## SET

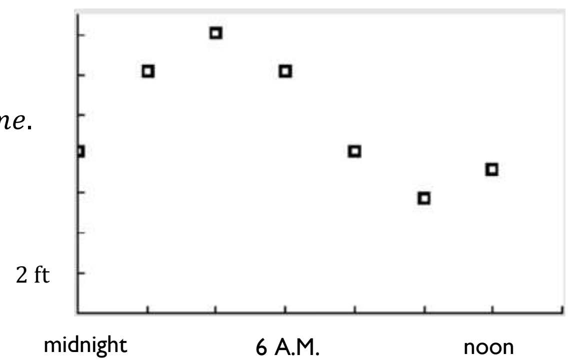
Topic: Mathematical modeling using sine and cosine functions

Many real-life situations such as sound waves, weather patterns, and electrical currents can be modeled by sine and cosine functions. The table below shows the depth of water (in feet) at the end of a wharf as it varies with the tides at various times during the morning.

| *t* (time) | midnight | 2 A.M. | 4 A.M. | 6 A.M. | 8 A.M. | 10 A.M. | noon |
|---|---|---|---|---|---|---|---|
| *d* (depth) | 8.16 | 12.16 | 14.08 | 12.16 | 8.16 | 5.76 | 7.26 |

We can use a trigonometric function to model the data. Suppose you choose cosine.
$y = A\cos(bt - c) + d, \text{where } y \text{ is depth at any time.}$

The *amplitude* will be the distance from the average of the highest and lowest values. This will be the average depth ($d$).




5. Sketch the line that shows the average depth.

6. Find the amplitude. $A = \frac{1}{2}(high - low)$

7. Find the period. $p = 2|low\ time - high\ time|$. Since a normal period for sine is 2π. The new period for our model will be $\frac{2\pi}{p}$ $so\ b = \frac{2\pi}{p}$. (Use the *p* you calculated, divide and turn it into a decimal.)

8. High tide occurred 4 hours after midnight. The formula for the displacement is $4 = \frac{c}{b}$. Use *b* and solve for *c.*

9. Now that you have your values for *A, b, c*, and *d*, put them into an equation.
$y = A\cos(bt - c) + d$

10. Use your model to calculate the depth at 9 A.M. and 3 P.M.

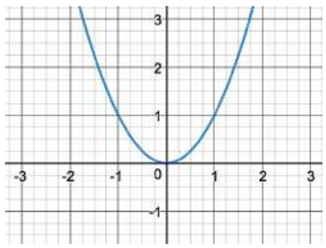
11. A boat needs at least 10 feet of water to dock at the wharf. During what interval of time in the afternoon can it safely dock?

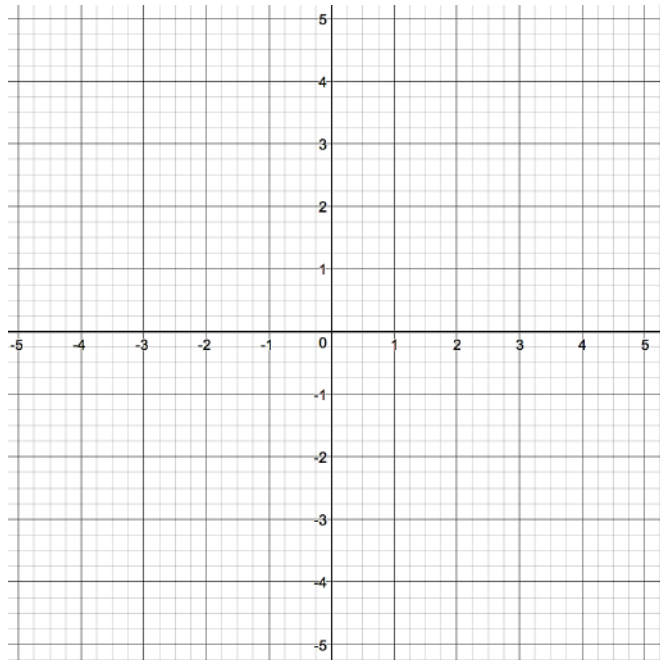
## GO

Topic: Connecting transformations on functions

**The equation and graph of a parent function is given. For each transformation, describe the change on the graph of the parent function. Then graph the functions on the same grid.**

12. $f(x) = x^2$



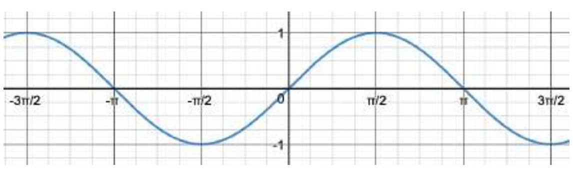




a. $f(x) = x^2 - 3$
Description:

b. $f(x) = (x-3)^2 - 4$
Description:

c. $f(x) = 2(x-3)^2 - 4$
Description:

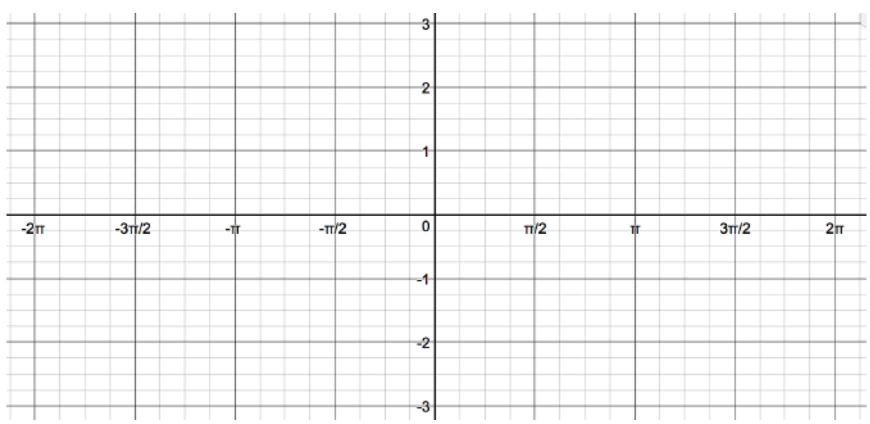
13. $g(x) = sinx$




a. $g(x) = (sinx) + 2$
Description:

b. $g(x) = sin\left(x + \frac{\pi}{2}\right) - 1$
Description:

c. $g(x) = 2sin\left(x + \frac{\pi}{2}\right) - 1$
Description:



# 7.3 Getting on the Right Wavelength

### *A Practice Understanding Task*

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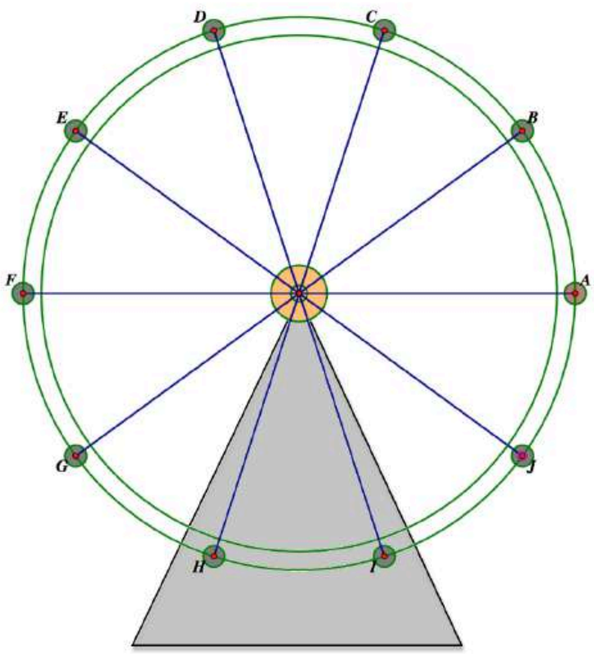
The Ferris wheel in the following diagram has a radius of 40 feet, its center is 50 feet from the ground, and it makes one revolution counterclockwise every 18 seconds.




1. Write the equation of the height of the rider at any time $t$, if at $t = 0$ the rider is at position A (Use radians to measure the angle of rotation).

2. At what time(s) is the rider 70 feet above the ground? Show the details of how you answered this question.

3. If you used a sine function in question 1, revise your equation to model the same motion with a cosine function. If you used a cosine function, revise your equation to model the motion with a sine function.

4. Write the equation of the height of the rider at any time $t$, if at $t$ = 0 the rider is at position D (Use radians to measure the angle of rotation).

5. For the equation you wrote in question 4, at what time(s) is the rider 80 feet above the ground? Show or explain the details of how you answered this question.

6. Choose any other starting position and write the equation of the height of the rider at any time $t$, if at $t$ = 0 the rider is at the position you chose. (Use radians to measure the angle of rotation). Also change other features of the Ferris wheel, such as the height of the center, the radius, the direction of rotation and/or the length of time for a single rotation. (Record your equation and description of your Ferris wheel here.)

7. Trade the equation you wrote in question 7 with a partner and see if he or she can determine the essential features of your Ferris wheel: height of center, radius, period of revolution, direction of revolution, starting position of the rider. Resolve any issues where you and your partner have differences in your descriptions of the Ferris wheel modeled by your equation.

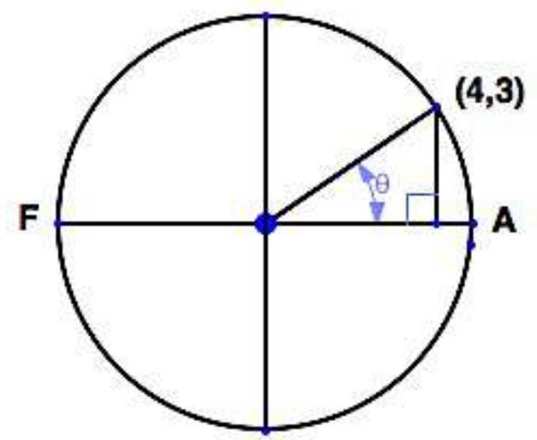
READY, SET, GO! Name Period Date

## READY

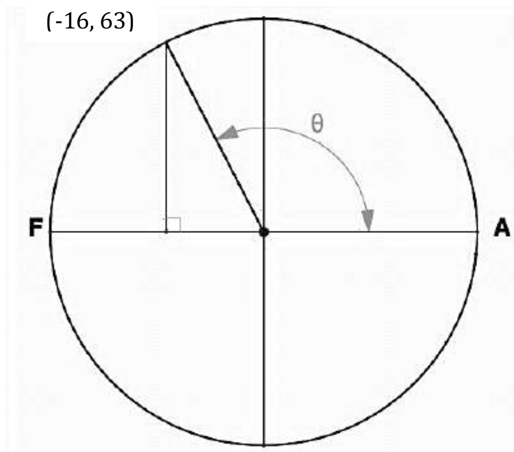
Topic: Using the definition of tangent

**Use what you know about the definition of tangent in a right triangle to find the value of tangent $\theta$ for each of the right triangles below.**

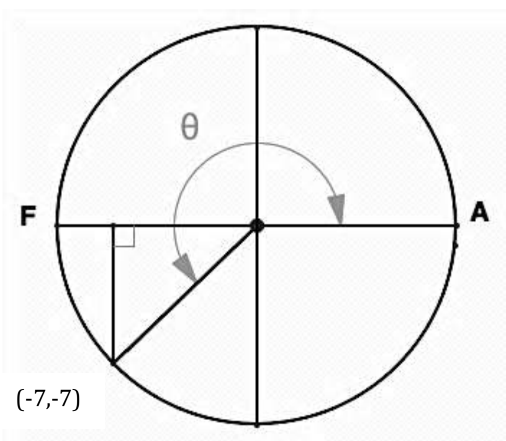
1. $\tan\theta =$



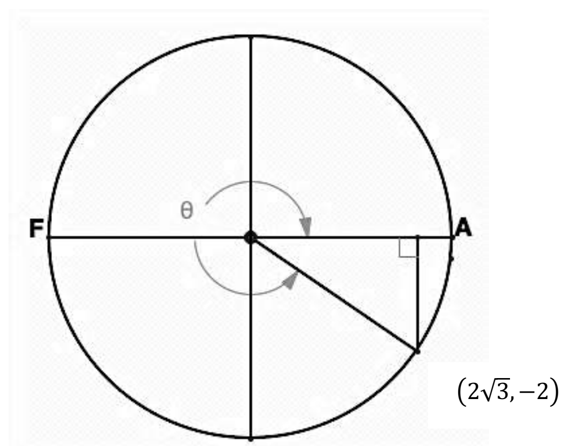

2. $\tan\theta =$




3. $\tan\theta =$




4. $\tan\theta =$



5. In each graph, the angle of rotation is indicated by an arc and $\theta$. Describe the angles of rotation from $0\ to\ 2\pi$ that make tangent $\theta$ be positive and the angles of rotation that make tangent $\theta$ be negative.

## SET

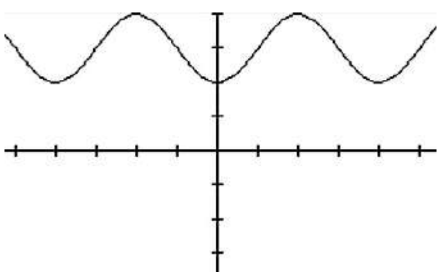
Topic: Connecting trig graphs with their equations

**Match each trigonometric representation on the left with an equivalent representation on the right. Then check your answers with a graphing calculator.** (The scale on the vertical axis is 1. The scale on the horizontal axis is $\left(\frac{\pi}{2}\right)$.)

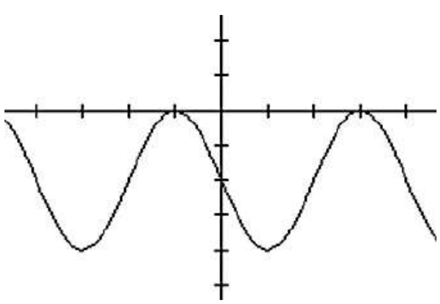
6. $y = -3sin\left(\theta + \frac{\pi}{2}\right)$

7. $y = 3\cos\left(\theta + \frac{\pi}{2}\right)$

8.



9.



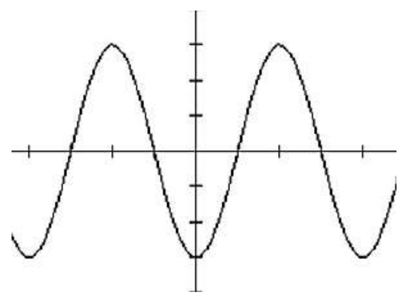
10. $y = \sin\left(2\left(\theta + \frac{\pi}{2}\right)\right) - 2$

11. $y = \sin(x + \pi)$

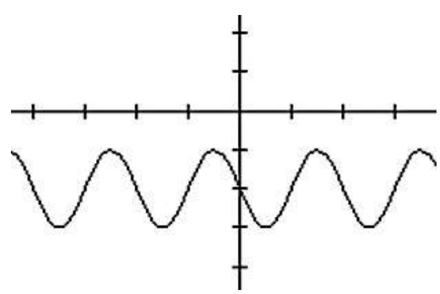
A. $y = -3\sin\theta$

B. $y = -sin\theta$

C.



D.



E. $y = 2cos\left(\theta + \frac{\pi}{2}\right) - 2$

F. $y = \cos(x + \pi) + 3$

12. Choose the equation(s) that has the same graph as $y = cos\theta$.

    a. $y = \cos(-\theta)$

    b. $y = \cos(\theta - \pi)$

    Use the unit circle to explain why they are the same.

13. Choose the equation(s) that has the same graph as $y = -sin\theta$.

    a. $y = \sin(\theta + \pi)$

    b. $y = \sin(\theta - \pi)$

    Use the unit circle to explain why they are the same.

**For each function, identify the amplitude, period, horizontal shift, and vertical shift.**

14. $f(t) = 150 \cos\left(\frac{\pi}{6}(t - 8)\right) + 80$

    amplitude:

    period:

    horizontal shift:

    vertical shift:

15. $f(t) = 4.5 \sin\left(\frac{\pi}{4}t + \frac{3}{4}\right) + 8$

    amplitude:

    period:

    horizontal shift:

    vertical shift:

## GO

Topic: Making sense of composite trig functions

*Recall that a composite function places one function such as g(x), inside the other, f(x), by replacing the x in f(x) with the entire function g(x). In general, the notation is* $f\left(g(x)\right)$. *Also, recall that inverse functions "undo" each other. Since,* $sin^{-1}\left(\frac{1}{2}\right)$ *is an angle of* $30°$ *because* $sin\, 30° = \frac{1}{2}$ *the composite* $sin\left(sin^{-1}\frac{1}{2}\right)$ *is simply* $\frac{1}{2}$. *Sine function "undoes" what* $\sin^{-1}\theta$ *was does.*

*Not all composite functions are inverses. Note: problems18 – 24.*

**Answer the following.**

16. $\sin\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$

17. $\cos\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$

18. $\tan(\tan^{-1} 9.52)$

19. $\sin\left(\cos^{-1}\frac{1}{2}\right)$

20. $\cos(\tan^{-1} 1)$

21. $\sin(\tan^{-1} 2.75)$

22. $\cos(\sin^{-1} 1)$

23. $\cos(\tan^{-1} 0)$

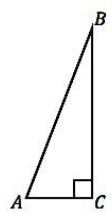
24. $\sin(\tan^{-1} undefined)$

# 7.4H Off on a Tangent

## *A Develop and Solidify Understanding Task*



Recall that the right triangle definition of the tangent ratio is:



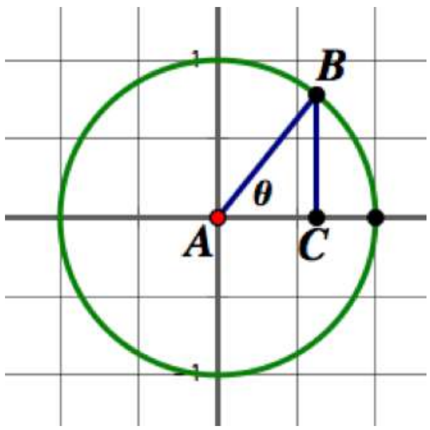

$$\tan(A) = \frac{\textit{length of side opposite angle } A}{\textit{length of side adjacent to angle } A}$$

1. Revise this definition to find the tangent of any angle of rotation, given in either radians or degrees. Explain why your definition is reasonable.

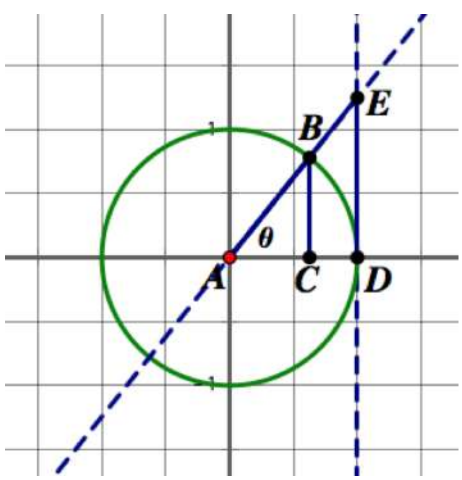
2. Revise this definition to find the tangent of any angle of rotation drawn in standard position on the unit circle. Explain why your definition is reasonable.

We have observed that on the unit circle the value of sine and cosine can be represented with the length of a line segment.

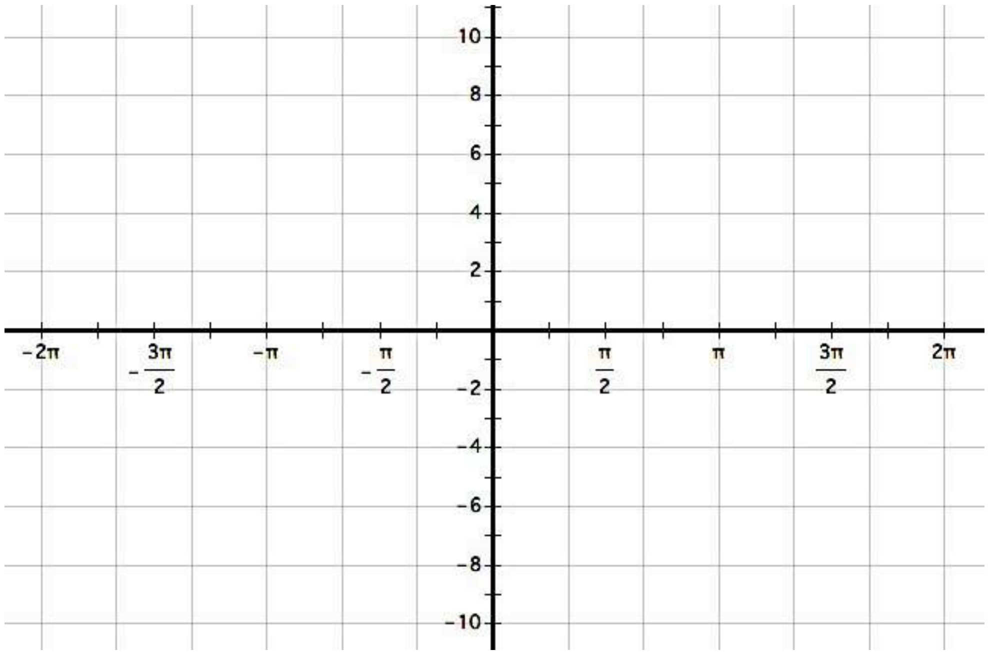
3. Indicate on the following diagram which segment's length represents the value of $\sin(\theta)$ and which represents the value of $\cos(\theta)$ for the given angle θ.




There is also a line segment that can be defined on the unit circle so that its length represents the value of $\tan(\theta)$. Consider the length of $\overline{DE}$ in the unit circle diagram below. Note that $\Delta ADE$ and $\Delta ABC$ are right triangles. Write a convincing argument explaining why the length of segment $DE$ is equivalent to the value of $\tan(\theta)$ for the given angle θ.



4. On the coordinate axes below sketch the graph of $y = \tan(\theta)$ by considering the length of segment $DE$ as θ rotates through angles from 0 radians to 2π radians. Explain any interesting features you notice in your graph.




Extend your graph of $y = \tan(\theta)$ by considering the length of segment $DE$ as θ rotates through negative angles from 0 radians to -2π radians.

5. Using your unit circle diagrams from the task *Water Wheels and the Unit Circle*, give exact values for the following trigonometric expressions:

a. $\tan\left(\frac{\pi}{6}\right) =$　　b. $\tan\left(\frac{5\pi}{6}\right) =$　　c. $\tan\left(\frac{7\pi}{6}\right) =$

d. $\tan\left(\frac{\pi}{4}\right) =$　　e. $\tan\left(\frac{3\pi}{4}\right) =$　　f. $\tan\left(\frac{11\pi}{6}\right) =$

g. $\tan\left(\frac{\pi}{2}\right) =$　　h. $\tan(\pi) =$　　i. $\tan\left(\frac{7\pi}{3}\right) =$

Functions are often classified based on the following definitions:

- A function $f(x)$ is classified as an **odd function** if $f(-\theta) = -f(\theta)$
- A function $f(x)$ is classified as an **even function** if $f(-\theta) = f(\theta)$

6. Based on these definitions and your work in this module, determine how to classify each of the following trigonometric functions.

- The function $y = sin(x)$ would be classified as an [odd function, even function, neither an odd or even function]. Give evidence for your response.

- The function $y = cos(x)$ would be classified as an [odd function, even function, neither an odd or even function]. Give evidence for your response.

- The function $y = tan(x)$ would be classified as an [odd function, even function, neither an odd or even function]. Give evidence for your response.

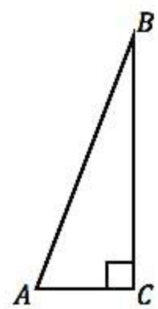
**Extra for Experts:**

When defining the trig ratios using right triangles we named possible ratios of sides, such as the sine ratio defined as the ratio of *the length of the side opposite the acute angle* to *the length of the hypotenuse*, the cosine ratio defined as the ratio of *the length of the side adjacent to the acute angle* to *the length of the hypotenuse*, and the tangent ratio defined as *the length of the side opposite the acute angle* to *the length of the side adjacent to the acute angle.*

It is sometimes useful to consider the reciprocals of these ratios, leading to the definition of three additional trignometric ratios: *secant, cosecant* and *cotangent,* as defined below.

**The secant ratio:**

$$\sec(A) = \frac{length\ of\ the\ hypotenuse}{length\ of\ side\ adjacent\ to\ angle\ A}$$




**The cosecant ratio:**

$$\csc(A) = \frac{length\ of\ the\ hypotenuse}{length\ of\ side\ opposite\ angle\ A}$$

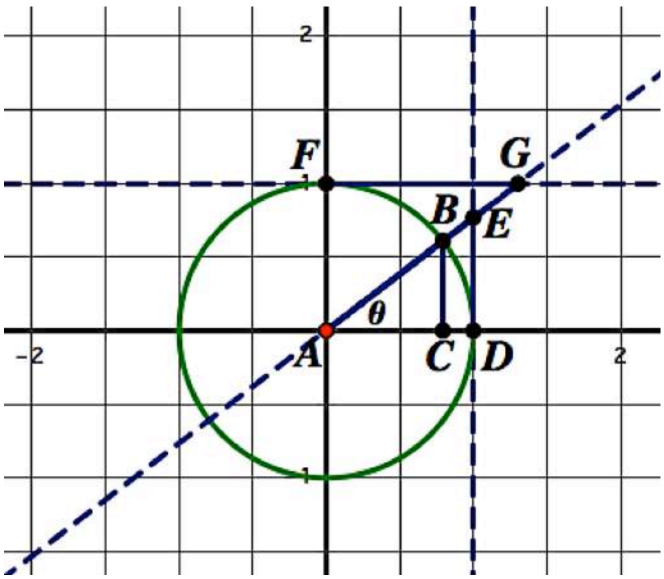
**The cotangent ratio:**

$$\cot(A) = \frac{length\ of\ side\ adjacent\ to\ angle\ A}{length\ of\ side\ opposite\ angle\ A}$$

7. Complete the following statements:

   a. The ____________ ratio is the reciprical of the sine ratio.
   b. The ____________ ratio is the reciprical of the cosine ratio.
   c. The ____________ ratio is the reciprical of the tangent ratio.
   d. $\frac{1}{\cos\theta} =$ $\qquad$ $\frac{1}{\sin\theta} =$ $\qquad$ $\frac{1}{\tan\theta} =$
   e. $\frac{1}{\cot\theta} =$ $\qquad$ $\frac{1}{\sec\theta} =$ $\qquad$ $\frac{1}{\csc\theta} =$

There are also line segments that can be defined on the unit circle so that their lengths represents the value of $\sec(\theta)$, $\csc(\theta)$, $or$ $\cot(\theta)$. Consider the lengths of $\overline{AE}, \overline{AG}$ $and$ $\overline{FG}$ in the unit circle diagram below. Note that $\Delta ADE$ and $\Delta AFG$ are right triangles and $\angle FGA \cong \angle DAG$ since they are alternate interior angles formed by parallel lines $\overleftrightarrow{FG}$ and $\overleftrightarrow{AD}$ intersected by transversal $\overleftrightarrow{AG}$.

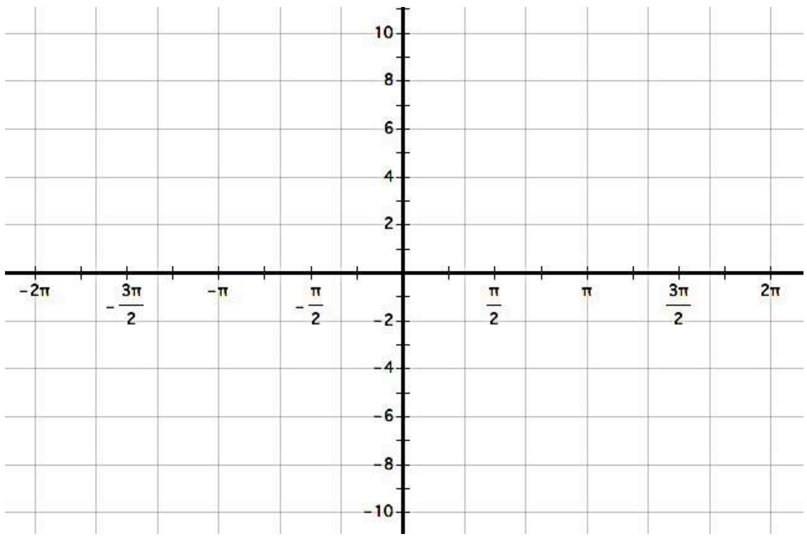



8. Which segment has a length that would be equal to $\sec(\theta)$? Explain how you know.

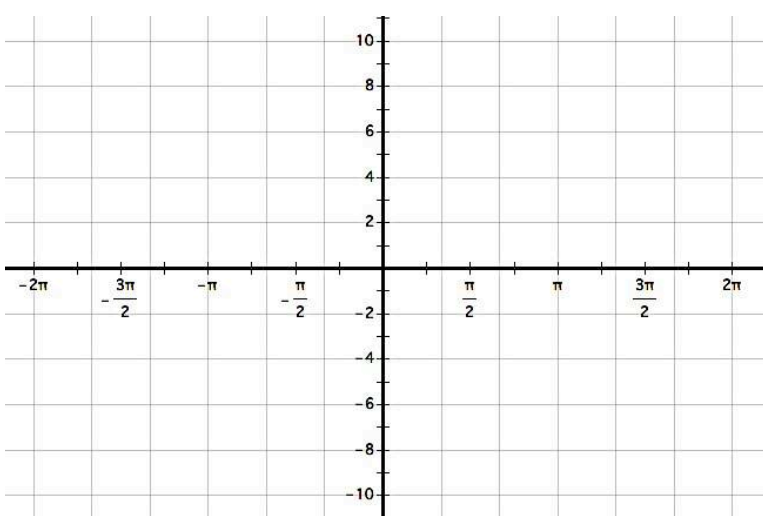
9. Which segment has a length that would be equal to $\csc(\theta)$? Explain how you know.

10. Which segment has a length that would be equal to $\cot(\theta)$? Explain how you know.

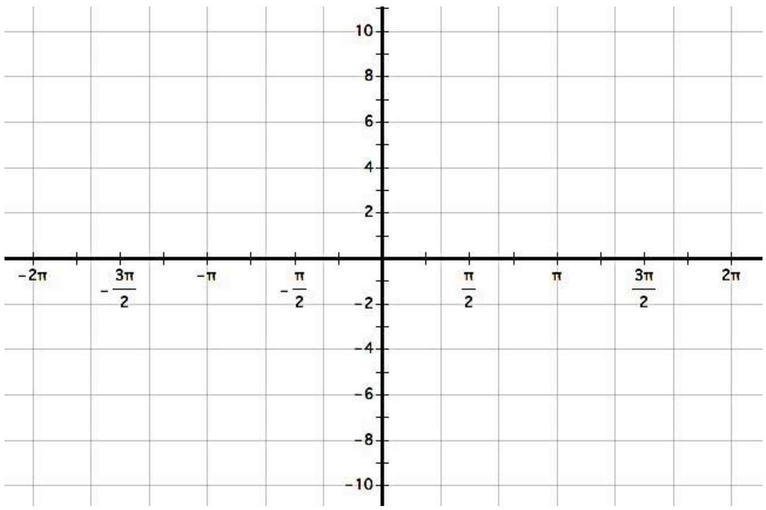
11. On the coordinate axes below sketch the graph of $y = \sec(\theta)$ by considering the length of its corresponding segment in the unit circle diagram above as $\theta$ rotates through angles from 0 radians to $2\pi$ radians, and from 0 radians to -$2\pi$ radians. Explain any interesting features you notice in your graph.



12. On the coordinate axes below sketch the graph of $y = \csc(\theta)$ by considering the length of its corresponding segment in the unit circle diagram above as θ rotates through angles from 0 radians to $2\pi$ radians, and from 0 radians to -$2\pi$ radians. Explain any interesting features you notice in your graph.



13. On the coordinate axes below sketch the graph of $y = \cot(\theta)$ by considering the length of its corresponding segment in the unit circle diagram above as θ rotates through angles from 0 radians to 2π radians, and from 0 radians to -2π radians. Explain any interesting features you notice in your graph.



READY, SET, GO! Name Period Date

## READY

Topic: Making rigid and non-rigid transformations on functions

**The equation of a parent function is given. Write a new equation with the given transformations. Then sketch the new function on the same graph as the parent function. (If the function has asymptotes, sketch them in.)**

1. $y = x^2$

Vertical shift: up 8

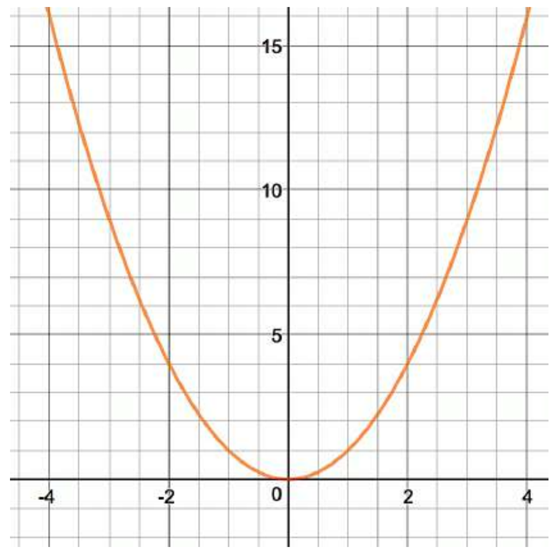
horizontal shift: left 3

dilation: ¼

Equation:

Domain:

Range:




2. $y = \frac{1}{x}$

Vertical shift: up 4

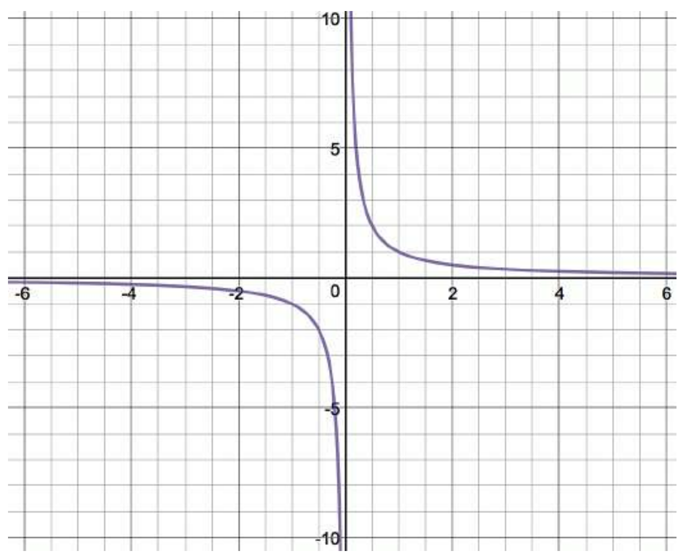
horizontal shift: right 3

dilation: $-1$

Equation:

Domain:

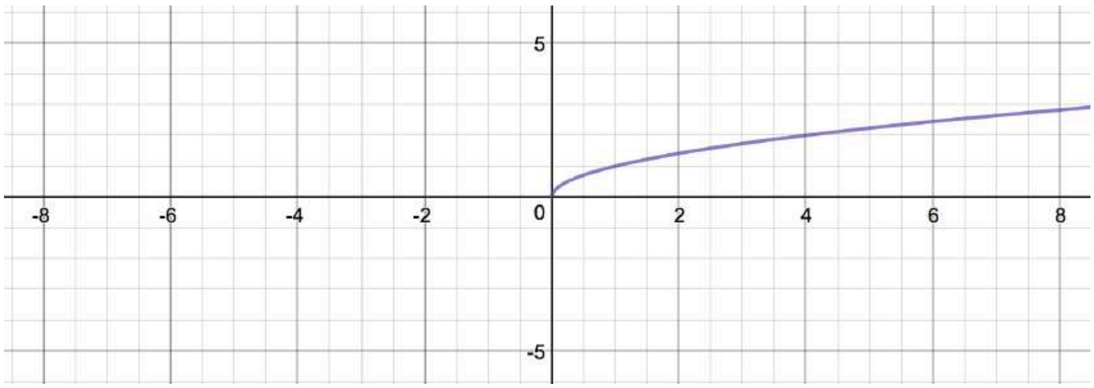
Range:



3. $y = \sqrt{x}$

Vertical shift: none. horizontal shift: left 5 dilation: 3

Equation: Domain: Range:




4. $y = \sin x$

Vertical shift: 1

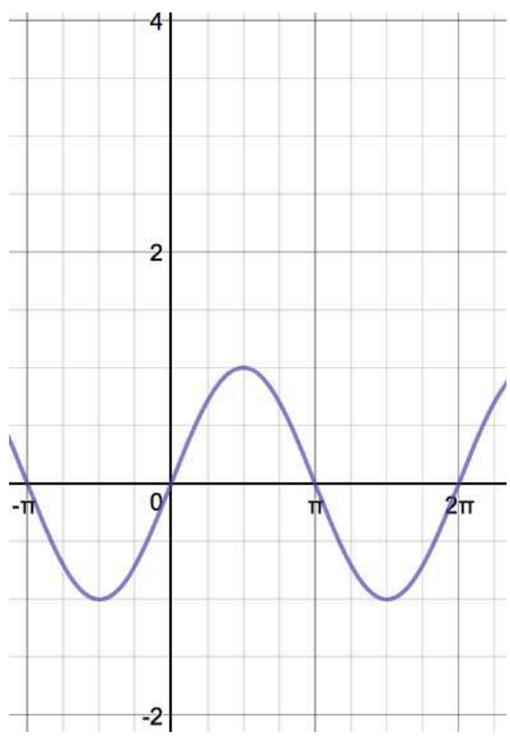
horizontal shift: left $\frac{\pi}{2}$

dilation (amplitude): 3

Equation:

Domain:

Range:

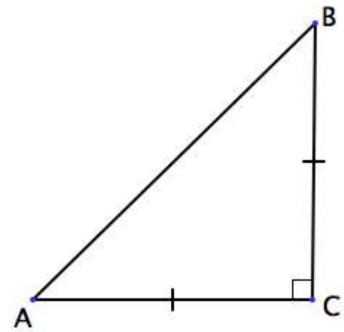


## SET

Topic: Connecting values in the special triangles with radian measures

5. Triangle ABC is a right triangle. AB = 1.

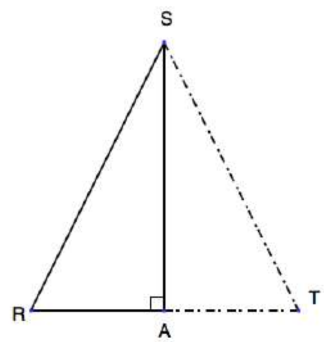
   Use the information in the figure to label the length of the sides and measure of the angles.




6. Triangle RST is an equilateral triangle. RS = 1
   $\overline{SA}$ is an altitude

   Use the information in the figure to label the length of the sides, the length of $\overline{RA}$ , and the *exact* length of $\overline{SA}$ .

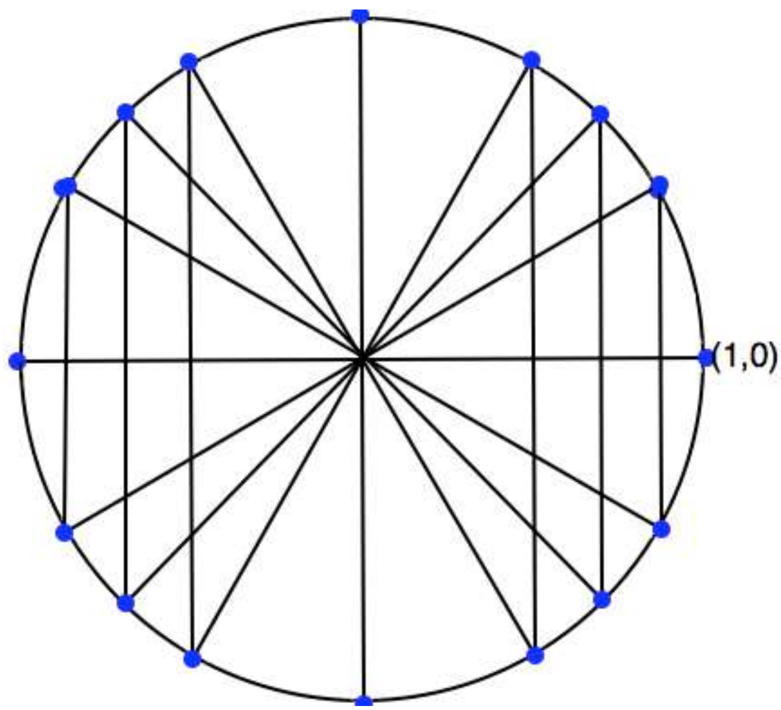
   Label the measure of angles RSA and SRA.



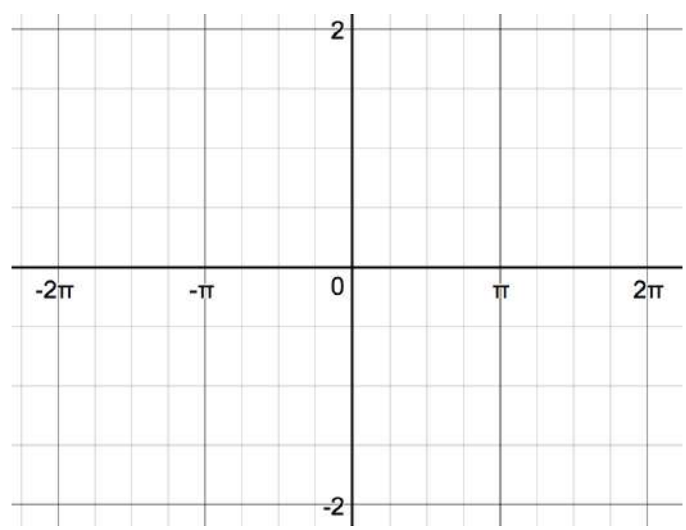

7. Use what you know about the unit circle and the information from the figures in problems 6 and 7 to fill in the table. Some values will be undefined.

| *function* | $\theta = \frac{\pi}{6}$ | $\theta = \frac{\pi}{4}$ | $\theta = \frac{\pi}{3}$ | $\theta = \frac{\pi}{2}$ | $\theta = \pi$ | $\theta = \frac{3\pi}{2}$ | $\theta = 2\pi$ |
|---|---|---|---|---|---|---|---|
| $\sin\theta$ | | | | | | | |
| $\cos\theta$ | | | | | | | |
| $\tan\theta$ | | | | | | | |
| $\csc\theta$ | | | | | | | |
| $\sec\theta$ | | | | | | | |
| $\cot\theta$ | | | | | | | |

8. Label all of the points and angles of rotation in the given unit circle.




9. **Graph $\boldsymbol{f(x) = \tan\theta}$.** Use your table of values above for $f(x) = \tan\theta$. Sketch your asymptotes with dotted lines.




10. Where do asymptotes always occur?

11. How can you use the sine function to determine the location of the asymptotes for cosecant?

## GO

Topic: Recalling trig facts

**Answer the questions below. Be sure you can justify your thinking.**

12. Given triangle ABC with angle C being a right angle, what is the sum of $m\angle A + m\angle B$ ?

13. Identify the quadrants in which $\sin\theta$ *is positive*.

14. Identify the quadrants in which $\cos\theta$ *is negative*.

15. Identify the quadrants in which $\tan\theta$ *is positive*.

16. Explain why it is impossible for $\sin\theta > 1$.

17. Name the angles of rotation (in radians) for when $\sin\theta = \cos\theta$.

18. For which trig functions do a positive rotation and a negative rotation always give the same value?

19. Explain why in the unit circle $\tan\theta = \frac{y}{x}$.

20. Which function gives the slope of the hypotenuse in a right triangle?

21. Explain why $\sin\theta = \cos(90° - \theta)$.

22. Write the Pythagorean Identity and then prove it.

23. Name the trigonometric function(s) that are even functions.

# 7.5 Maintaining Your Identity

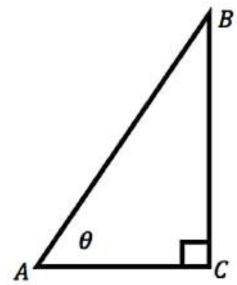
## *A Develop Understanding Task*




Right triangles and the unit circle provide images that can be used to derive, explain and justify a variety of trigonometric identities.

1. For example, how might the right triangle diagram at the right help you justify why the following identity is true for all angles $\theta$ between 0° and 90°?

   $\sin(\theta) = \cos(90° - \theta)$

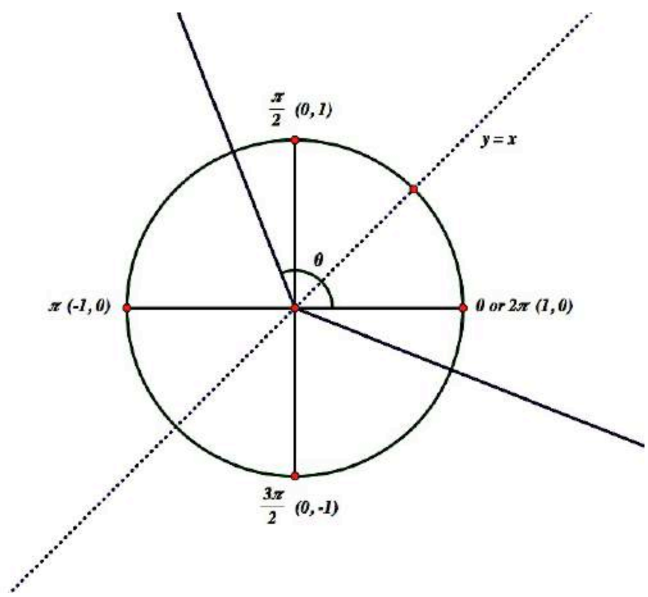



2. Since we have extended our definition of the sine to include angles of rotation, rather than just the acute angles in a right triangle, we might wonder if this identity is true for all angles $\theta$, not just those that measure between 0° and 90°?

   A version of this identity that uses radian rather than degree measure would look like this:

   $$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$

   How might you use this unit circle diagram to justify why this identity is true for all angles $\theta$?



**Fundamental Trig Identities**

3. Here are some additional trig identities. Use either a right triangle diagram or a unit circle diagram to justify why each is true.

   a. $\sin(-\theta) = -\sin(\theta)$

   b. $\cos(-\theta) = \cos(\theta)$

   c. $\sin^2\theta + \cos^2\theta = 1$ [Note: This is the preferred notation for $(\sin\theta)^2 + (\cos\theta)^2 = 1$]

   d. $\frac{\sin\theta}{\cos\theta} = \tan\theta$

4. Use right triangles or a unit circle to help you form a conjecture for how to complete the following statements as trig identities. How might you use graphs to gain additional supporting evidence that your conjectures are true?

a. $\sin(\pi - \theta) =$ ____________________ and $\cos(\pi - \theta) =$ ____________________

b. $\sin(\pi + \theta) =$ ____________________ and $\cos(\pi + \theta) =$ ____________________

c. $\sin(2\pi - \theta) =$ __________________ and $\cos(2\pi - \theta) =$ __________________

5. We can use algebra, along with some fundamental trig identities, to prove other identities. For example, how can you use algebra and the identities listed above to prove the following identities?

a. $\tan(-\theta) = -\tan(\theta)$

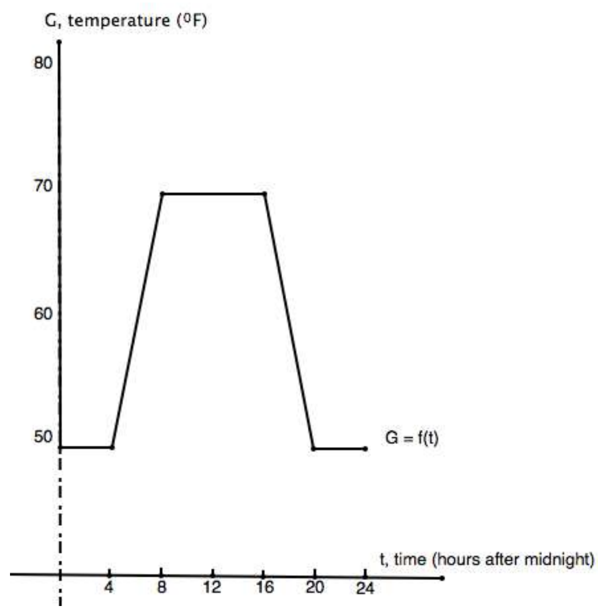
b. $\tan(\pi + \theta) = \tan(\theta)$

READY, SET, GO! Name Period Date

## READY

Topic: Modeling transformations

A school building is kept warm only during school hours, in order to save money. Figure 6.1 shows a graph of the temperature, G, in °F, as a function of time, t, in hours after midnight. At midnight (t = 0), the building's temperature is 50 °F. This temperature remains the same until 4 AM. Then the heater begins to warm the building so that by 8 am the temperature is 70°F. That temperature is maintained until 4 pm, when the building begins to cool. By 8 pm, the temperature has returned to 50°F and will remain at that temperature until 4 am.




1. In January many students are sick with the flu. The custodian decides to keep the building 5 °F warmer. Sketch the graph of the new schedule on figure 6.1.

2. If $G = f(t)$ is the function that describes the original temperature setting, what would be the function for the January setting?

3. In the spring, the drill team begins early morning practice. The custodian then changes the original setting to start 2 hours earlier The building now begins to warm at 2 am instead of 4 am and reaches 70 °F at 6 am. It begins cooling off at 2 pm instead of 4 pm and returns to 50 °F at 6 pm instead of 8 pm. Sketch the graph of the new schedule on figure 6.1.

4. If $G = f(t)$ is the function that describes the original temperature setting, what would be the function for the spring setting?

## SET

Topic: Using trigonometric identities to find additional trig values

The *Cofunction identity* states: $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ *and* $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$

**Complete the statements, using the** *Cofunction identity.*

5. Sin 70° = cos _____°

6. Sin 28° = cos _____°

7. Cos 54° = sin _____°

8. Sin 9° = cos _____°

9. Cos 72° = sin _____°

10. Cos 45° = sin _____°

11. $\text{Cos}\,\frac{\pi}{8} = \sin$ _____

12. $\text{Sin}\,\frac{5\pi}{12} = \cos$ _____

13. $\text{Sin}\,\frac{3\pi}{10} = \cos$ _____

14. Let $\sin\theta = \frac{3}{4}$.

   a) Use the *Pythagorean identity* $(sin^2\theta + cos^2\theta = 1)$, to find the value of $\cos\theta$.

   b) Use the *Quotient identity* $\left(\tan\theta = \frac{\sin\theta}{\cos\theta}\right)$, the given information, and your answer in part (a) to calculate the value of $\tan\theta$.

15. Let $\cos\beta = \frac{12}{13}$.

   a) Find $\sin\beta$. Use the *Pythagorean identity* $(sin^2\theta + cos^2\theta = 1)$.

   b) Find $\tan\beta$. Use the *Quotient identity* $\left(\tan\theta = \frac{\sin\theta}{\cos\theta}\right)$.

   c) Find $\cos\left(\frac{\pi}{2} - \theta\right)$. Use a *Cofunction identity.*

**Use trigonometric definitions and identities to prove the statements below.**

16. $\tan\theta\cos\theta = \sin\theta$

17. $(1+\cos\beta)(1-\cos\beta) = \sin^2\beta$

18. $(1+\sin\alpha)(1-\sin\alpha) = \cos^2\alpha$

19. $sin^2W - cos^2W = 2sin^2W - 1$

## GO

Topic: Solving simple trig equations using the special angle relationships

**Find <u>two</u> solutions of the equation. Give your answers in degrees $(0° \leq \theta \leq 360°)$ and radians $(0 \leq \theta \leq 2\pi)$. Do NOT use a calculator.**

20. $\sin\theta = \frac{1}{2}$

degrees: ______________________________

radians: ______________________________

21. $\sin\theta = -\frac{1}{2}$

degrees: ______________________________

radians: ______________________________

22. $\cos\theta = \frac{\sqrt{2}}{2}$

degrees: ______________________________

radians: ______________________________

23. $\sin\theta = -\frac{\sqrt{3}}{2}$

degrees: ______________________________

radians: ______________________________

24. $\tan\theta = -1$

degrees: ______________________________

radians: ______________________________

25. $\tan\theta = \sqrt{3}$

degrees: ______________________________

radians: ______________________________

# 7.6 Hidden Identities

## *A Practice Understanding Task*



Note: Because trig functions are periodic, trig equations often have multiple solutions. Typically, we are only interested in the solutions that lie within a restricted interval, usually the interval from 0 to $2\pi$. In this task you should find all solutions to the trig equations that occur on $[0, 2\pi]$.

To sharpen their trig skills, Alyce, Javier and Veronica are trying to learn how to solve some trig equations in a math refresher text that they found in an old trunk one of the adults had brought to the archeological site. Here is how each of them thought about one of the problems:

$$\textit{Solve}\text{: } \cos\left(\frac{\pi}{2} - \theta\right) = \frac{1}{2}$$

Alyce: I used the inverse cosine function.

Javier: I first used an identity, and then an inverse trig function. But it was not the same inverse trig function that Alyce used.

Veronica: I graphed $y_1 = \cos\left(\frac{\pi}{2} - \theta\right)$ and $y_2 = \frac{1}{2}$ on my calculator. I seem to have found more solutions.

1. Using their statements as clues, go back and solve the equation the way that each of the friends did.

2. How does Veronica's solutions match with Alyce and Javier's? What might be different?

Solve each of the following trig equations by adapting Alyce and Javier's strategies: that is, you may want to see if the equation can be simplified using one of the trig identities you learned in the previous task; and once you have isolated a trig function on one side of the equation, you can undo that trig function by taking the inverse trig function on both sides of the equation. Once you have a solution, you may want to check to see if you have found all possible solutions on the interval $[0, 2\pi]$ by using a graph as shown in Veronica's strategy.

3. $\sin(-\theta) = -\frac{1}{2}$

4. $\cos\theta \cdot \tan\theta = \frac{\sqrt{3}}{2}$

5. $\sin(2\theta) = \frac{\sqrt{2}}{2}$

READY, SET, GO! Name Period Date

## READY

Topic: Using the calculator to find angles of rotation

**Use your calculator and what you know about where sine and cosine are positive and negative in the unit circle to find the <u>two angles</u> that are solutions to the equation. Make sure $\theta$ *is always* $0 < \theta \leq 2\pi$. Round your answers to 4 decimals.**
*(Your calculator should be set in radians.)*

*You will notice that your calculator will sometimes give you a negative angle. That is because the calculator is programmed to restrict the angle of rotation so that the inverse of the function is also a function. Since the requested answers have been restricted to positive rotations, if the calculator gives you a negative angle of rotation, you will need to figure out the positive coterminal angle for the angle that your calculator has given you.*

1. $\sin\theta = \frac{4}{5}$
2. $\sin\theta = \frac{2}{7}$
3. $\sin\theta = -\frac{1}{10}$
4. $\sin\theta = -\frac{13}{14}$

5. $\cos\theta = \frac{11}{12}$
6. $\cos\theta = \frac{1}{9}$
7. $\cos\theta = -\frac{7}{8}$
8. $\cos\theta = -\frac{2}{5}$

*Note: When you ask your calculator for the angle, you are "undoing" the trig function. Finding the angle is finding the inverse trig function. When you see "***Find** $\sin^{-1}\left(\frac{4}{5}\right)$" *, you are being asked to find the angle that makes it true. The answer would be the same as the answer your calculator gave you in #1. Another notation that means the inverse sine function is* **arcsin** $\left(\frac{4}{5}\right)$.

## SET

Topic: Verifying trig identities with tables, unit circles, and graphs

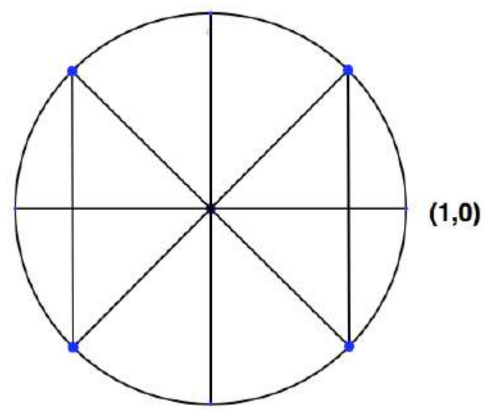
9. **Use the values in the table to verify the Pythagorean identity** $\left(\sin^2\theta + \cos^2\theta = 1\right)$**. Then use the Quotient Identity** $\left(\tan\theta = \frac{\sin\theta}{\cos\theta}\right)$ **and the values in the table to write the value of tangent** $\theta$**.**

| **9.** | $sin\,\theta$ | $cos\,\theta$ | $sin^2\theta + cos^2\theta = 1$ | $tan\theta$ |
|---|---|---|---|---|
| a. | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | | |
| b. | $\frac{-3}{5}$ | $\frac{4}{5}$ | | |
| c. | $-\frac{5}{13}$ | $-\frac{12}{13}$ | | |
| d. | $\frac{\sqrt{14}}{7}$ | $\frac{\sqrt{35}}{7}$ | | |
| e. | $-1$ | $0$ | | |
| f. | $0$ | $1$ | | |

10. Label the angles of rotation and the coordinate points around the unit circle on the right. Then use these points to help you fill in the blank.

$cos(\pi - \theta) =$ ______________________

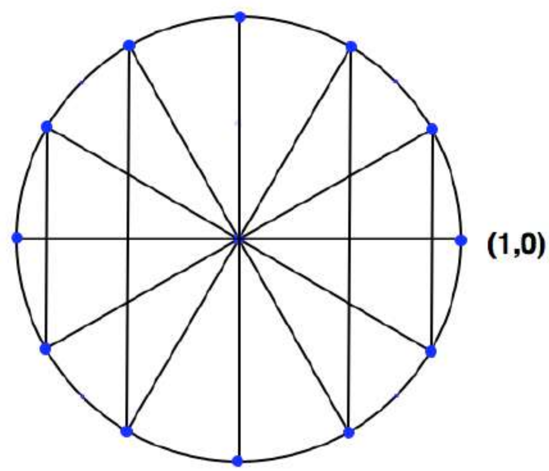
Make your thinking visible by using the diagram. Explain your reasoning.



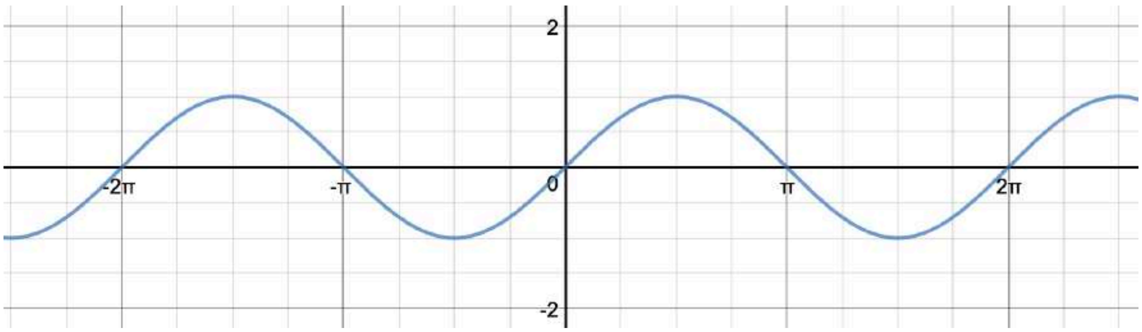

11. Label the angles of rotation and the coordinate points around the unit circle on the right. Then use these points to help you fill in the blank.

$sin(\pi + \theta) =$ ______________________

Make your thinking visible by using the diagram. Explain your reasoning.



12. Use the graph of $\sin\theta$ to help you fill in the blank. $sin(2\pi - \theta) =$ ____________________




Make your thinking visible by using the graph. Explain your reasoning.

## GO

Topic: Finding the central angle when given arc length and radius

**Find the radian measure of the central angle of a circle of radius *r* that intercepts an arc of length *s*.** $(\boldsymbol{s = r\theta})$

**Round answers to 4 decimal places.**

| *Radius* | *Arc Length* | *Angle measure in radians* |
|---|---|---|
| 13. 35 mm | 11 mm | |
| 14. 14 feet | 9 feet | |
| 15. 16.5 m | 28 m | |
| 16. 45 miles | 90 miles | |

# 7.7H Double Identity

## *A Solidify Understanding Task*



**Sum and Difference Identities**

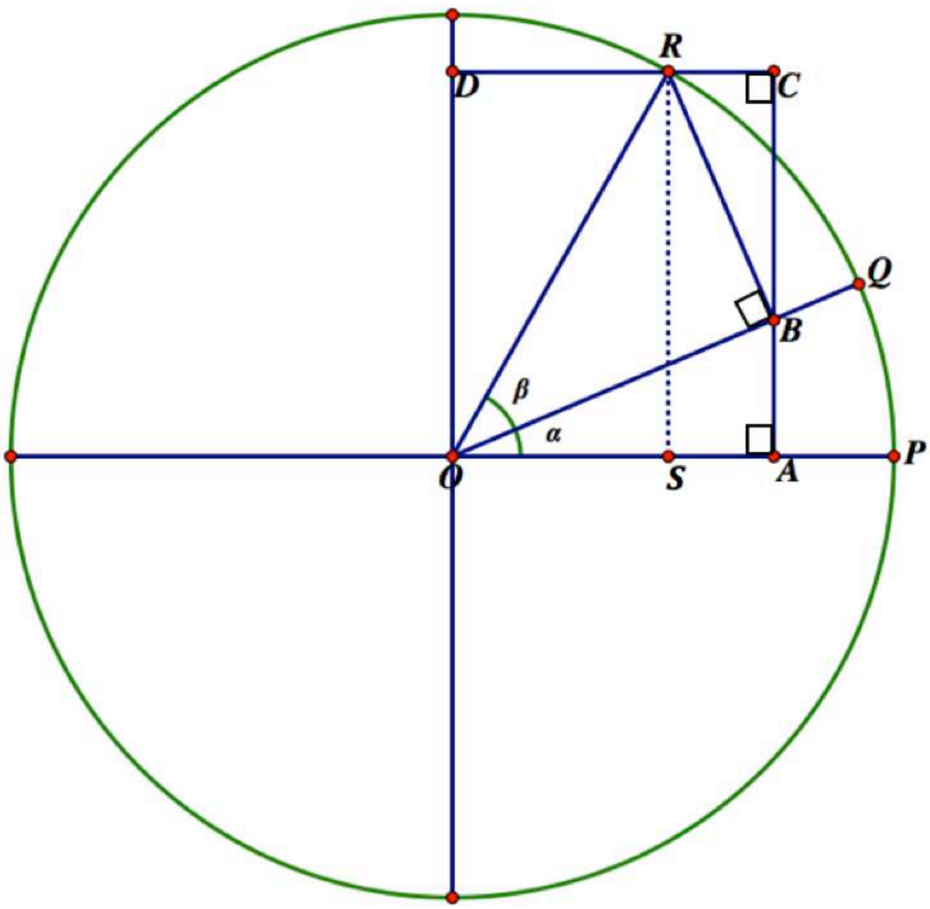
Sometimes it is useful to be able to find the sine and cosine of an angle that is the sum of two consecutive angles of rotation. In the diagram below, point $P$ has been rotated $\alpha$ radians counterclockwise around the unit circle to point $Q$, and then point $Q$ has been rotated an additional $\beta$ radians counterclockwise to point $R$. In this task you will examine how the sine and cosine of angle $\alpha$, angle $\beta$ and the sum of the two angles, angle $\alpha + \beta$, are related?

1. Do you think this is a true statement: $\sin(\alpha+\beta) = \sin\alpha + \sin\beta$

   Why or why not?

Examine the diagram on the following page. Figure $OACD$ is a rectangle. Can you use this diagram to state a true relationship that completes this identity? (Your teacher has some hint cards if you need them, but the basic idea is to label all of the segments on the sides of rectangle $OACD$ using right triangle trig relationships.) Based on congruent line segments labeled with trigonometric measures in the diagram, complete the following identity:

2. $\sin(\alpha+\beta) =$ ________________________________




Once you have an identity for $\sin(\alpha + \beta)$ you can find an identity for $\sin(\alpha - \beta)$ algebraically. Begin by noting that $\sin(\alpha - \beta) = \sin(\alpha + (-\beta))$ and apply the identity you found in question #2, along with the identities you explored previously: $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$.

3. $\sin(\alpha - \beta) =$ ____________________________________

You can find an identity for $\cos(\alpha + \beta)$ in the diagram above also. Since $\overline{OA} \cong \overline{DC}$, and $DC = DR + RC$, using trigonometry to determine the lengths of segments $OA$, $DR$ and $RC$ will reveal this relationship. (Again, your teacher has hint cards if you need them.)

4. $\cos(\alpha + \beta) =$ ________________________________________

Now you can also complete this identity using reasoning similar to what you did in question #3.

5. $\cos(\alpha - \beta) =$ ________________________________________

The following identities are known as the double angle identities, but they are just special cases of the sum identities you found above.

6. $\sin(2\alpha) = \sin(\alpha + \alpha) =$ ________________________________________

7. $\cos(2\alpha) = \cos(\alpha + \alpha) =$ ________________________________________

READY, SET, GO! Name Period Date

## READY

Topic: Solving equations

**Solve for x.**

1. $2x - 1 = -2$
2. $6x - 3 = \sqrt{2}$
3. $x^2 - 3 = -2$
4. $4x^2 = 3$
5. How many solutions are there for #1 and #2?
6. How many solutions are there for #3 and #4?
7. $2(sinx) - 1 = -2$
8. $6(cosx) - 3 = \sqrt{2}$
9. $(sin)^2x - 3 = -2$
10. $4(cos)^2x = 3$
11. How many solutions are there for #7 and #8?
12. Problems 7 and 8 are very similar to problems 1 and 2. Why do 7 and 8 have more answers?
13. How many solutions are there for problems 9 and 10?
14. Problems 9 and 10 are very similar to problems 3 and 4. Why do 9 and 10 have more answers?

## SET

Topic: Exploring the sum and difference identities

**The sum and difference formulas are given below.**

| | |
|---|---|
| $sin(u + v) = (sinu)(cosv) + (cosu)(sinv)$ | $cos(u + v) = (cosu)(cosv) - (sinu)(sinv)$ |
| $sin(u - v) = (sinu)(cosv) - (cosu)(sinv)$ | $cos(u - v) = (cosu)(cosv) + (sinu)(sinv)$ |

The sum and difference formulas can be used to find **exact** values of trigonometric functions involving sums or differences of special angles. The hard part is recognizing that an unfamiliar angle might be the sum or difference of an angle you know.

**Write the following as the sum or difference of two angles for which you know the trig values.**

15. 75°
16. 345°
17. $\frac{\pi}{12}$
18. $-\frac{13\pi}{12}$

**Find the exact value of the following using the sum and difference formulas.**

19. $sin 75°$
20. $cos 345°$
21. $cos\frac{\pi}{12}$
22. $sin\left(-\frac{13\pi}{12}\right)$

**Find the exact value of the function given that $\boldsymbol{sin\, u = \frac{5}{13}}$ *and* $\boldsymbol{cosv = -\frac{3}{5}}$.** (Both u and v are in Q. 2.)

23. $sin(u+v)$
24. $cos(u+v)$

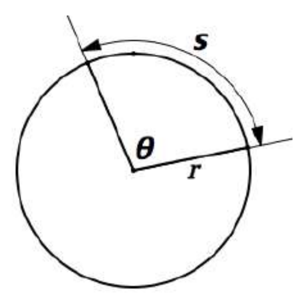
**Find the exact value of the function given that $\boldsymbol{sin\, u = -\frac{7}{25}}$ *and* $\boldsymbol{cosv = -\frac{4}{5}}$.** ( u and v are in Q. 3.)

25. $sin(u-v)$
26. $cos(u-v)$

## GO

Topic: Finding arc length

**Recall the formula for arc length: $\boldsymbol{s = r\theta}$, where $\boldsymbol{\theta}$ is always in radians.**
Write your answers with π in it. Then use your calculator to find the approximate length of the arc to 2 decimal places.




27. Find the length of an arc given that r = 10 in and $\theta = \frac{\pi}{4}$.
28. Find the arc length given r = 4 cm and $\theta = \frac{5\pi}{6}$.
29. Find the arc length given r = 72.0 ft and $\theta = \frac{\pi}{8}$.

# 7.8H The Amazing Inverse Trig Function Race

### *A Solidify Understanding Task*



To entertain themselves on weekends at the archeological dig, Javier has invented a game called "Find My Stake." The game consists of drawing two cards, one from a deck of cards that Javier calls "The Angle Specification" cards, and the other from a deck Javier calls the "Location" cards. Based on these two clues, Veronica and Alyce race to locate the position of the stake. The friend who finds the correct location first, wins a prize. Alyce wonders why they need to have two clues. Veronica wonders if two clues will always be enough.

With a partner, play Javier's game a few times using the two decks of cards that will be provided by your teacher. One of you will draw an "Angle Specification Card." The other will draw a "Location Card." See if you can determine the exact location of the stake that is described by the two clues given on the cards. Note that "Angle Specification" cards do not state an angle directly. Rather, they give information about the angle being specified, such as an inverse trig function statement or an equation to be solved. The "Location" cards give additional information to assist you in locating the stake, such as giving the $x$ or $y$-coordinate of the stake (but not both); or giving $r$, the distance from the central tower; or perhaps telling the quadrant in which the stake is located.

The archelogical site is laid out using both a rectangular grid system and a circular grid system. In the rectangular grid system the horizontal axis represents distances east and west of the central tower, and the vertical axis represents distances north or south of the central tower, the same as on a conventional map. In the circular grid system, concentric circles surround the central tower at equally spaced intervals. Javier has provided both a rectangular grid map and a circular grid map of the archeological site for Veronica and Alyce to use while playing the game. Likewise, your teacher will provide you with both types of grids as you play the game.

**Playing the Game**

With your partner, play the game at least three times as described above. For each time you play the game, do the following:

- Record the two clues you draw, one from each deck
- Show all work, including calcuations, that you do in an attempt to locate the stake
- Choose a rectangular grid or circular grid on which you will record the location of the stake—if you cannot locate the stake exactly, show all possible locations of the stake on the grid; if the clues provide contradictory information, state that a location is impossible to determine
- If possible, determine the location of the stake on both the rectangular and circular grids

1. Recall that Veronica wondered if two clues would be enough to locate the stake. After playing the game a few times, what do you think?

**Analyzing the Game**

Examine the clues given to you in the two decks of cards, and then do the following:

- Select a pair of cards that would determine a specific location for the stake—record the clues on the cards and explain why they determine a single, unique location
- Select another pair of cards that would suggest the stake can be located in more than one location—record the clues on the cards and explain why the location of the stake is not uniquely specified
- Select a third pair of cards that give contradictory information—record the clues on the cards and explain the conflict

Repeat these steps a few times until you can answer the following question.

2. *In general, what types of combinations of clues specify a unique location?*

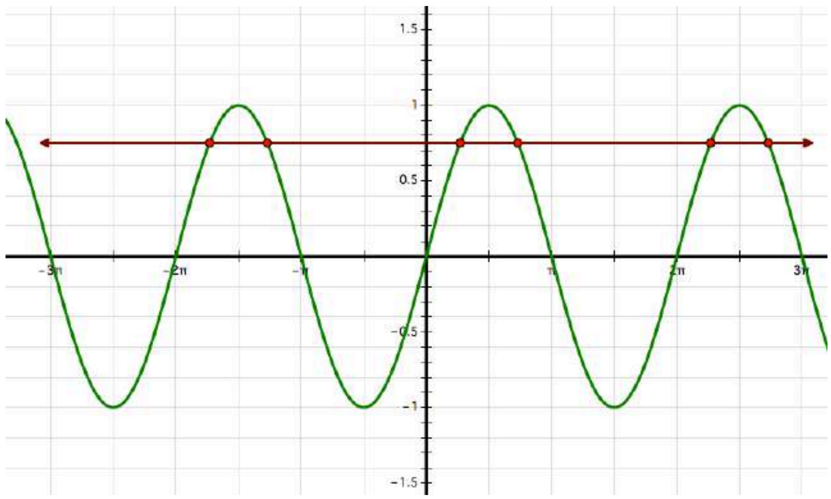
**Explaining the Game**

For each of the "Angle Specification" cards you had to answer the question, "What angle could fit this given information?" Perhaps you thought about the unit circle or used a calculator to answer this question. For angles of rotation, there are many answers to this question. Therefore, this question—by itself—does not define an inverse trig function.

Suppose you draw this clue from the set of "Angle Specification" cards:

$$\sin \theta = 0.75$$

Using your calculator, $\sin^{-1}(0.75) = 0.848$ radians. However, the following graph indicates other values of $\theta$ for which $\sin \theta = 0.75$.




3. Without tracing the graph or using any other calculator analysis tools, use the fact that $\sin^{-1}(0.75) = 0.848$ radians to find at least three other angles $\theta$ where $\sin \theta = 0.75$. (Each of these points shows up as a point of intersection between the sine curve and the line $y = 0.75$ in the graph shown above.)

Your calculator has been programmed to use the following definition for the inverse sine function, so that each time we find ***sin⁻¹*** of a number, we will get a unique solution.

***<u>Definition of the inverse sine function</u>*:** **$y = \sin^{-1} x$ means, "find the angle $y$,**

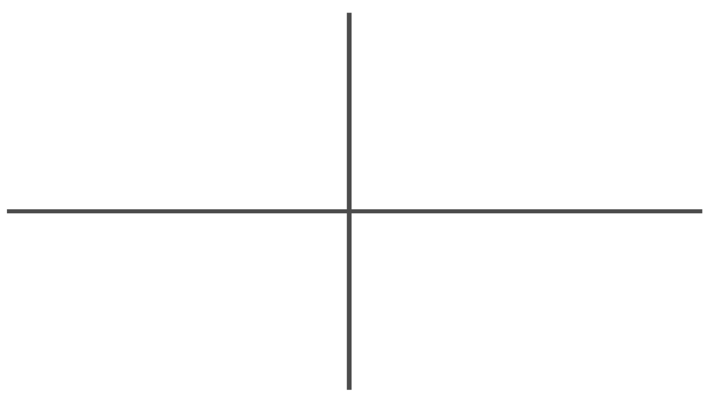
***on the interval*** $-\pi/2 \le y \le \pi/2$ **, such that $\sin y = x$."**

4. Based on the graph of the sine function, explain why defining the inverse trig function in this way guarantees that it will have a single, unique output.

5. Based on this definition, what is the domain of the inverse trig function?

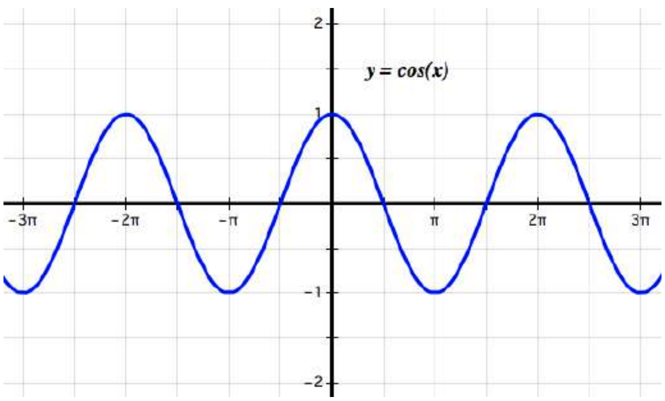
6. Based on this definition, what is the range of the inverse trig function?

7. Sketch a graph of the inverse sine function.

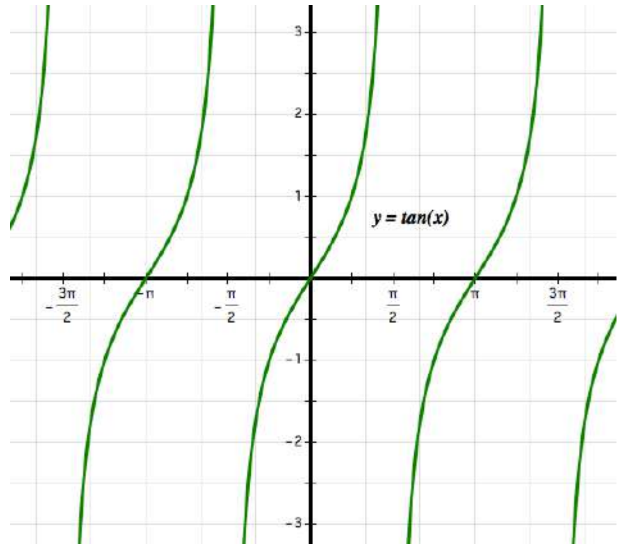


8. Suppose you draw this clue from the set of "Angle Specification" cards: $\sin\theta = -\frac{1}{2}$. What is the exact answer to this inverse sine expression: $\sin^{-1}(-\frac{1}{2})$ ?

Examine the graphs of the cosine function and the tangent function given below. How would you restrict the domains of these trig functions so that the inverse cosine function and the inverse tangent function can be constructed?




9.




9. Complete the definitions of the inverse cosine function and the inverse tangent function below. State the domain and range of each function, and sketch its graph.

*<u>Definition of the inverse cosine function</u>*:

Domain: Range:

Graph:

*<u>Definition of the inverse tangent function</u>*:

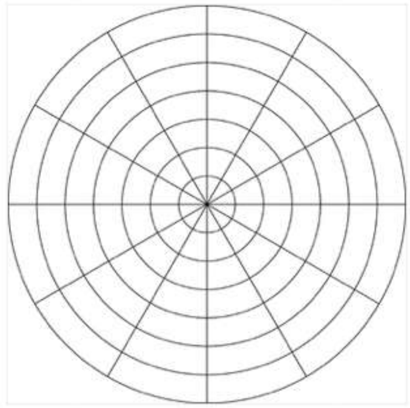
Domain: Range:

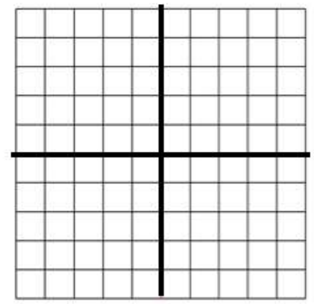
Graph:

*Game 1*

Location clue:

Angle clue:

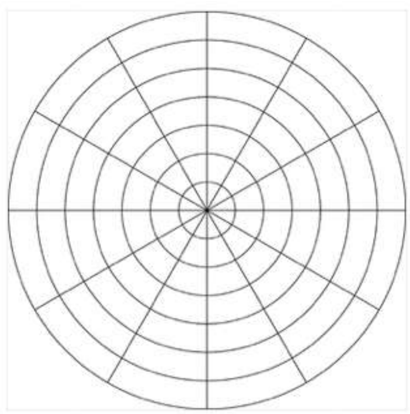


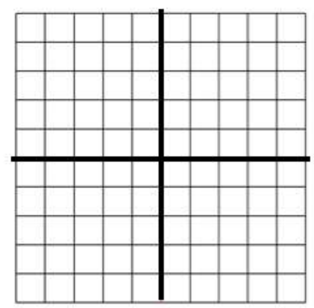


*Game 2*

Location clue:

Angle clue:

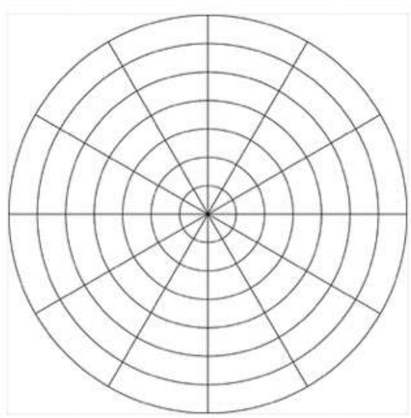


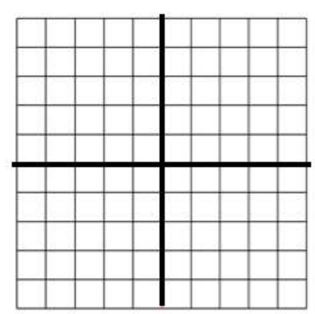


*Game 3*

Location clue:

Angle clue:





READY, SET, GO! Name Period Date

## READY

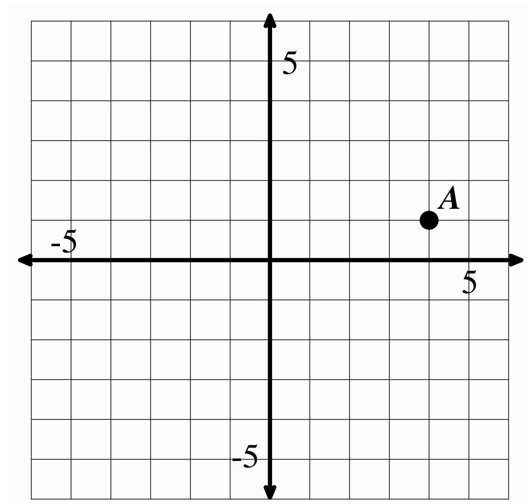
Topic: Recalling transformations using geometric notation

**Transform point A as indicated below.**

1. a. Apply the rule $(x, y) \to (x - 2, y - 5)$ to point A.
   Label as A'

   b. Apply the rule $(x, y) \to (x - 1, y + 3)$ to point A'.
   Label as A''

   c. Apply the rule $(x, y) \to (-2x, y)$ to point A'' and
   Label A'''

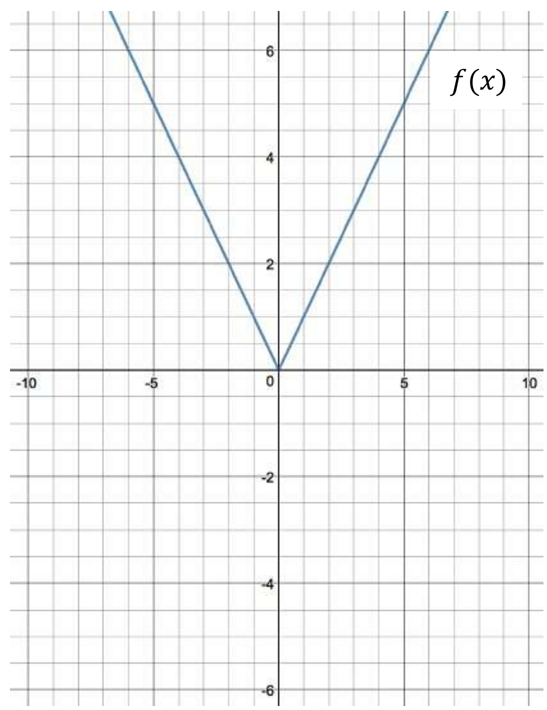



**Transform the given graph as indicated below.**

2. a. Apply the rule $(x, y) \to (x - 3, y)\ to\ f(x)$.
   Label $f'(x)$.

   b. Apply the rule $(x, y) \to (x, y - 5)\ to\ f(x)$.
   Label $f''(x)$.

   c. Apply the rule $(x, y) \to \left(x, \frac{1}{2}y\right)\ to\ f(x)$.
   Label $f'''(x)$.



## SET

Topic: Finding a specific angle of rotation using inverse trig functions

**Use the given information to find the missing angle $(\mathbf{0 \leq \theta \leq 2\pi})$.**
**Round answers to thousandths place (3 decimal places).**

3. $\cos\theta = 0.9848$; $\sin\theta > 0$

4. $\sin\theta = 0.9925$; $\tan\theta < 0$

5. $\cos\theta = 0.0872$; $\theta$ *is in Quadrant IV*

6. $\tan\theta = 0.3839$; $\cos\theta < 0$

7. $\cos\theta = 0$; $\sin\theta > 0$

8. $\sin\theta = -0.1908$; $\tan\theta > 0$

9. $\tan\theta = -0.4663$; $\sin\theta > 0$

10. $\tan\theta = -0.4663$; $\cos\theta > 0$

11. $\tan\theta = -1$; $\sin\theta > 0$

12. $\sin\theta = -1$

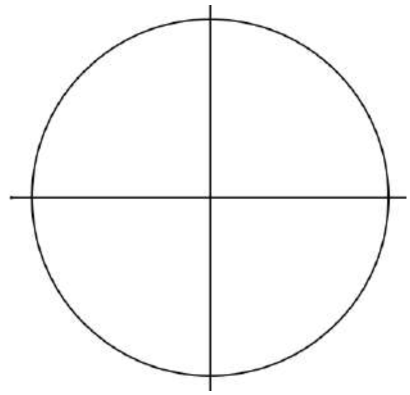
13. Explain why #12 needed only 1 clue to determine a unique value for $\theta$, and #3 – 11 required at least 2 clues.

## GO

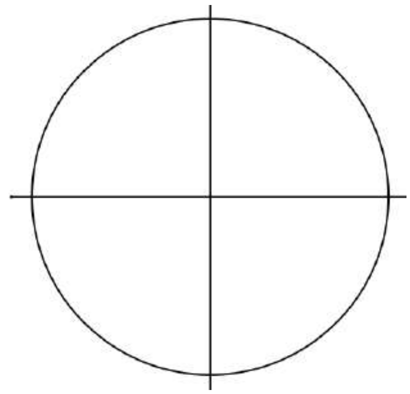
Topic: Calculating the area of a sector

**Find the area of a sector of a circle having radius $r$ and central angle $\theta$. $\left(A = \frac{1}{2}r^2\theta\right)$ Make a sketch and shade in the sector. Round answers to 2 decimals.**

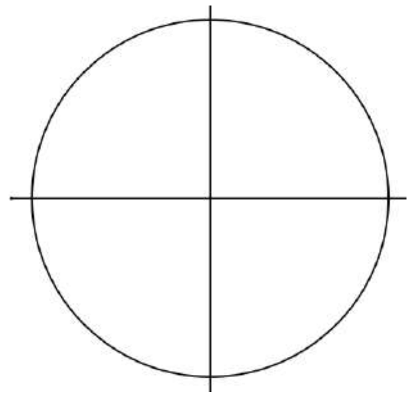
14. $r = 15\,m;\ \theta = \frac{2\pi}{5}\ radians$



15. $r = 29.2\,m;\ \theta = \frac{5\pi}{6}\ radians$



16. $r = 32\,mm;\ \theta = \frac{5\pi}{4}\ radians$



# 7.9H More Hidden Identities

## *A Practice Understanding Task*



Note: Because trig functions are periodic, trig equations generally have multiple solutions. In this task you should find all solutions to the trig equations by representing a sequence of solutions using the form $answer \pm k\pi n$, where $k\pi$ represents the interval between successive solutions.

In *Hidden Identities,* Javier observed that he might need to change the form of the trig expressions in a trig equation before he could use an inverse trig function to solve the equation. Changing the form of a trig expression involves looking for trig identities that apply to the expression. Sometimes you have to manipulate the expression algebraically before you can identify a trig identity that might apply.

Here is a synopsis of how Javier would have solved a problem in *Hidden Identities.* You can use his strategy to solve the remaining problems in this task.

1. Solve: $2\sin(x)\cos(x) + \frac{\sqrt{3}}{2} = \sqrt{3}$

**Idea #1:** Do you see a trig identity? If so, write the equation in an equivalent form using the identity. If not, can you manipulate the trig equation algebraically?

a. Which approach do you think Javier would need to take for his first step to solve this problem: use an identity or manipulate the equation algebraically?

b. Apply your thought:

**Idea #2**: Repeat idea #1 until you have isolated the trig function on one side of the equation.

c. Apply idea #2:

**Idea #3**: Apply the inverse trig function to both sides of the equation, which will give an equation of the form: *an expression* = *an angle;* solve this equation for the variable.

d. Apply idea #3:

**Idea #4**: This process finds one of an infinite number of solutions. Since trig functions are periodic, adding or subtracting multiple increments of the period will generate new solutions. Write this infinite set of solutions in the form $answer \pm k\pi n$, where $k\pi$ represents the interval between successive solutions.

e. Apply idea #4:

**Idea #5**: Since the inverse trig function used in #3 *is a function*, we only get one unique value as a solution to the inverse trig expression, while there may be another solution in a different quadrant on the unit circle. Find this alternate solution, and the corresponding infinite set of solutions to the equation.

f. Apply idea #5:

**Hints for solving trig equations**: Graphs of trig functions and the unit circle can be used as tools to help reason about the solutions to a trig equation.

g. Illustrate how a graph and the unit circle can help you find the solutions to this equation:

Solve the following trig equations by applying the ideas and hints described above.

2. $2\cos(3x) = \sqrt{2}$

3. $\sin^2(x) - \sin(x) - 2 = 0$

4. $2\cos^2(x) - \sqrt{3}\cos(x) = 0$

5. $(\sin(x) + \cos(x))^2 = \frac{3}{2}$

READY, SET, GO! Name Period Date

## READY

Topic: Examining forms of linear and quadratic functions

**The different forms of linear and quadratic functions are listed below. Explain how the structure of each form gives you information about the graph of the function.**

**Linear:**

1. Standard form: $ax + by = c$

2. Slope-intercept form: $y = mx + b$

3. Point-slope form: $y = m(x - x_1) + y_1$

**Quadratic:**

4. Standard form: $y = ax^2 + bx + c$

5. Factored form: $y = a(x - r_1)(x - r_2)$

6. Vertex form: $y = a(x - h)^2 + k$

## SET

Topic: Solving trigonometric equations

**Solve the following trig equations. Write your answer(s) in the form $\pm k\pi n$. where $k\pi$ represents the interval between successive solutions.**

7. $5cosx + \sqrt{2} = \sqrt{2}$

8. $2sinx + \sqrt{2} = 0$

9. $tanx + 2\sqrt{3} = 3\sqrt{3}$

10. $sin^2x = 3cos^2x$

11. $2 - sin^2x = 1 + cosx$

12. $2sin^2x + 3sinx + 1 = 0$

13. $2sin^2(2x) = 1$

14. $2cos^2x - cosx = 1$

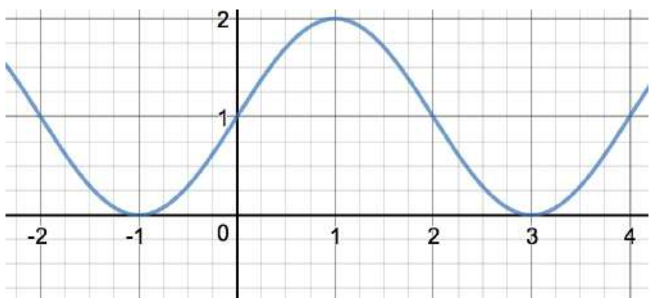
15. $2cos^2x - 5cosx + 2 = 0$

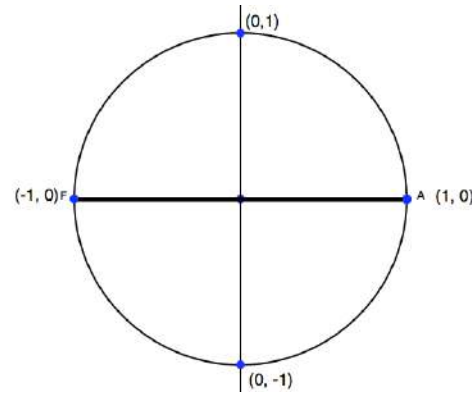
## GO

Topic: Interpreting graphs

**Use the graph to find all of the values for x when y=0 for the given equation. Write your answer(s) in the form $\pm k\pi n$. where $k\pi$ represents the interval between successive solutions.**

15. $y = sin\left(\frac{\pi x}{2}\right) + 1$

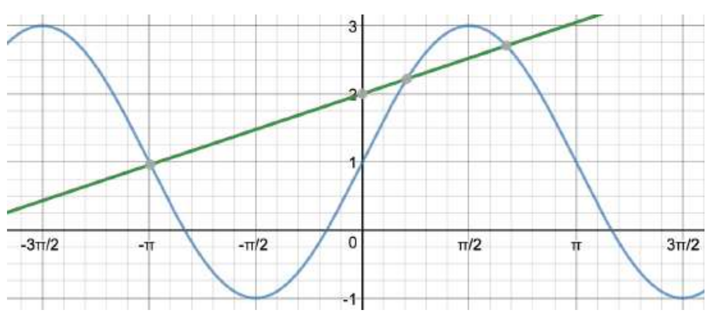



16. Use the unit circle to explain the solutions you found in problem 15.

**Use the graph to approximate the points of intersection of the graphs of $y_1$ *and* $y_2$.**

17. $y_1 = 2sinx + 1$
$y_2 = \frac{1}{3}x + 2$




18. The scale on the x-axis in the graph of problem 15 is 1. The scale in the graph of problem 17 is π/2. Yet the units on both axes is radians.

A.) Label the graph in #15 with the approximate location of $\frac{\pi}{2}$ *and* $\pi$.

B.) Label the graph in #17 with the approximate location of 1, 2, 3, and 4.

# 7.10H Polar Planes

## *A Develop Understanding Task*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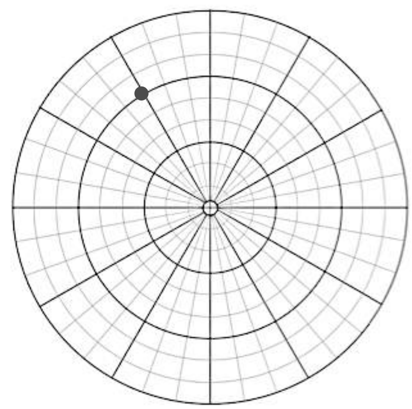
Alyce, Javier and Veronica have two different ways of recording the location of artifacts at the archeological dig: one way is to use **rectangular coordinates** $(x, y)$ and the other is to use **polar coordinates** $(r, \theta)$. The three friends know a lot about plotting points and graphing functions on a rectangular coordinate grid, and they are wondering if they can sketch graphs on a polar grid. They have found some polar graph paper in the archeological supplies, and are trying to make sense of it. They have learned that angles are measured with the initial ray pointing horizontally to the right (the positive horizontal axis), and positive angles are measured counterclockwise.



Javier thinks the location of the point plotted on the polar grid at the right is given by the polar coordinates (6, 120°).

Alyce thinks the location of the point is (6, -240°).

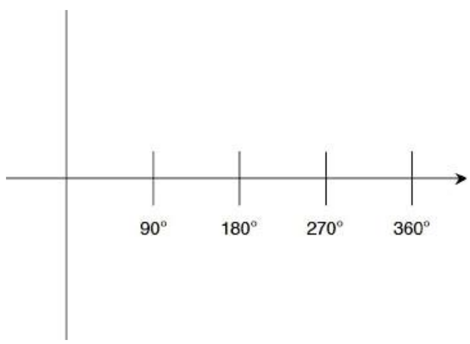
Veronica thinks the location of the point is (-6, 300°).

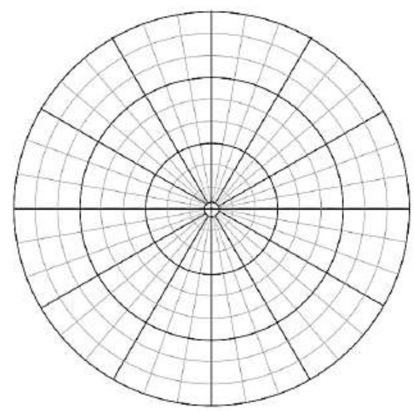
1. What do you think? Who has named the location of the point correctly? Explain why?

2. What are the rectangular coordinates of the plotted point?

Javier knows how to sketch the graph of $y = 6\sin(2\theta)$ on a rectangular grid, and wonders what the graph of $r = 6\sin(2\theta)$ might look like on a polar grid. When graphing trigonometric functions on a rectangular grid, Javier looks for key points, like maximums, minimums and $x$- and $y$-intercepts. He wonders how these strategies might work on a polar grid. Of course, he knows he can always create a table of values, plot a few consecutive points, and then connect them with a curve if he needs to.

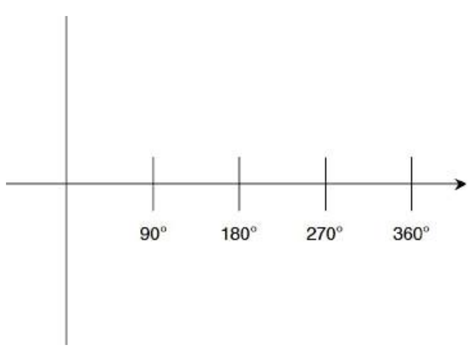
3. Sketch a graph of $y = 6\sin(2\theta)$ on the rectangular grid below and a graph of $r = 6\sin(2\theta)$ on the polar grid.

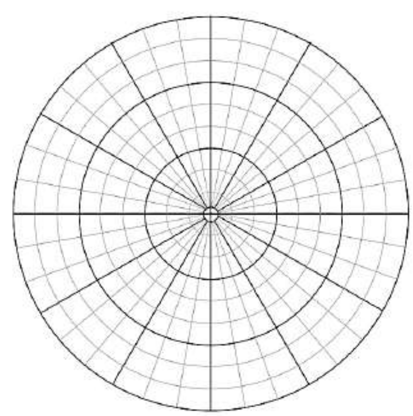


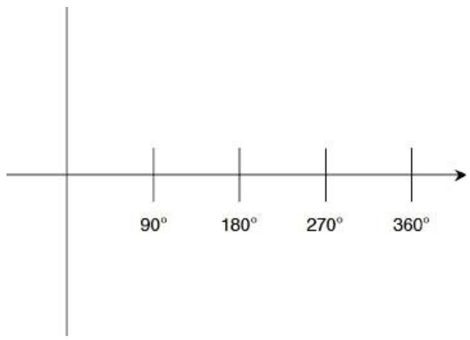
Javier wants to continue to compare the graphs of similar curves written in rectangular and polar form.

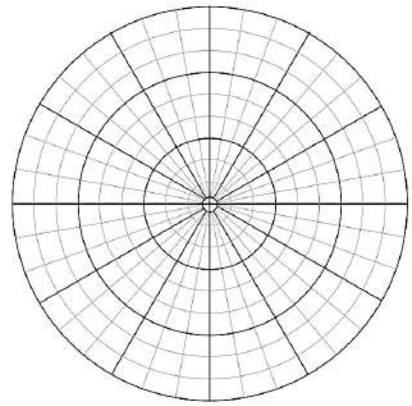
4. Graph $y = 6\sin(3\theta)$ and $r = 6\sin(3\theta)$ on the rectangular and polar grids.



5. Graph $y = 2 + 4\sin(\theta)$ and $r = 2 + 4\sin(\theta)$ on the rectangular and polar grids.



6. What strategies seem to work when graphing curves on a polar grid, so you don't have to plot a lot of individual points? Describe as many strategies as you can.

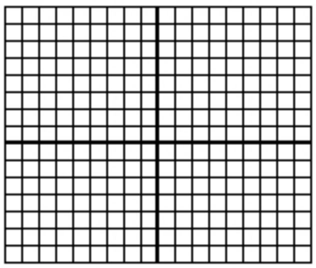
READY, SET, GO! Name Period Date

## READY

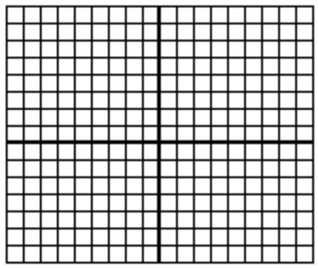
Topic: Applying the distance formula

**Find the distance from the origin (0, 0) to the given point in the rectangular plane.**

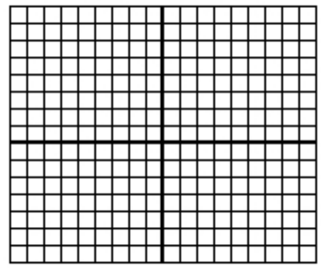
1. A (8, 6)



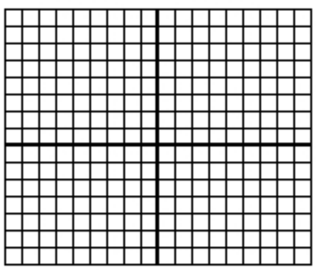
2. P (-5, -6)



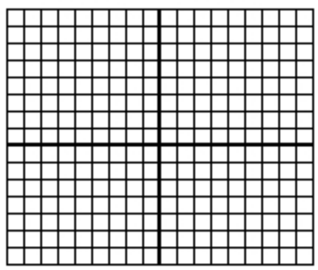
3. F (-7, 7)



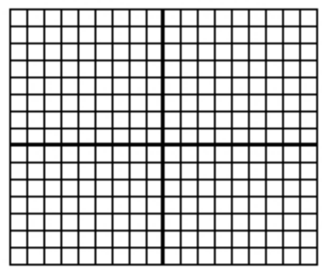
4. R (3, -4)



5. $G\left(\sqrt{3}, 1\right)$



6. $Q\left(3, \sqrt{7}\right)$

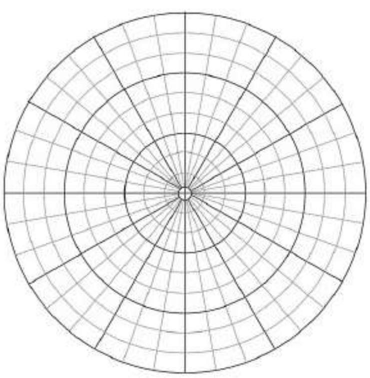


## SET

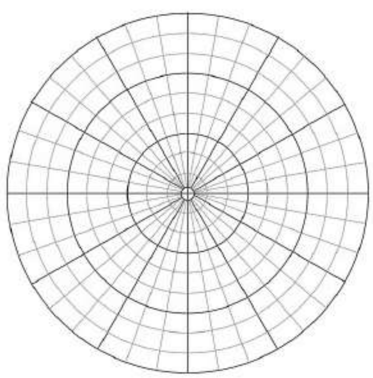
Topic: Graphing polar coordinates

**Plot the given point $(r, \theta)$ in the Polar Coordinate System.**

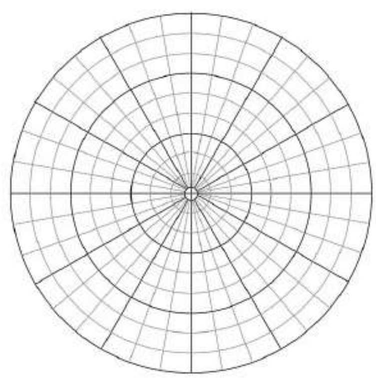
7. $(\boldsymbol{r}, \boldsymbol{\theta}) = \left(\mathbf{5}, \frac{\mathbf{2\pi}}{\mathbf{3}}\right)$



8. $(\boldsymbol{r}, \boldsymbol{\theta}) = \left(\mathbf{3}, \frac{\mathbf{7\pi}}{\mathbf{6}}\right)$



9. $(\boldsymbol{r}, \boldsymbol{\theta}) = \left(\mathbf{8}, \frac{\mathbf{5\pi}}{\mathbf{3}}\right)$



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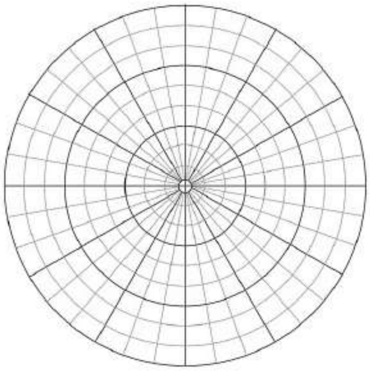
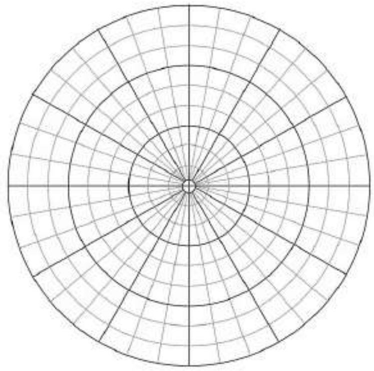
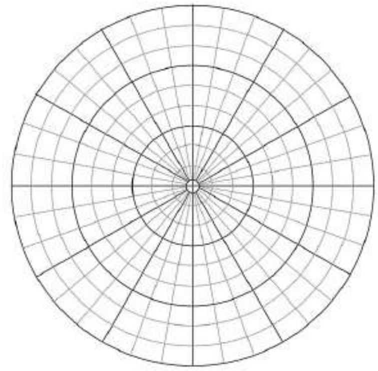
10. $(\boldsymbol{r}, \boldsymbol{\theta}) = \left(\mathbf{9},\ \frac{\boldsymbol{\pi}}{\mathbf{3}}\right)$



11. $(\boldsymbol{r}, \boldsymbol{\theta}) = \left(\mathbf{7},\ \frac{\mathbf{5}\boldsymbol{\pi}}{\mathbf{6}}\right)$

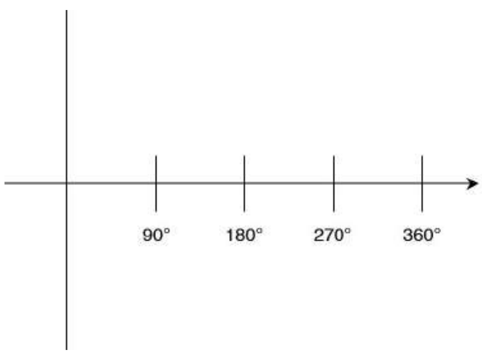


12. $(\boldsymbol{r}, \boldsymbol{\theta}) = \left(\mathbf{-6},\ \frac{\boldsymbol{\pi}}{\mathbf{3}}\right)$

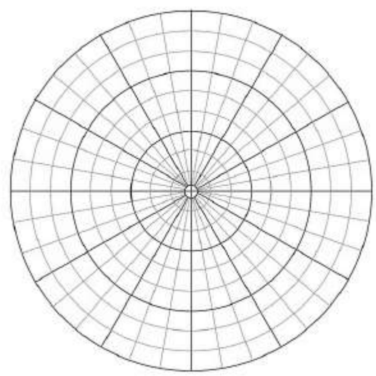


**Sketch the indicated graphs.**

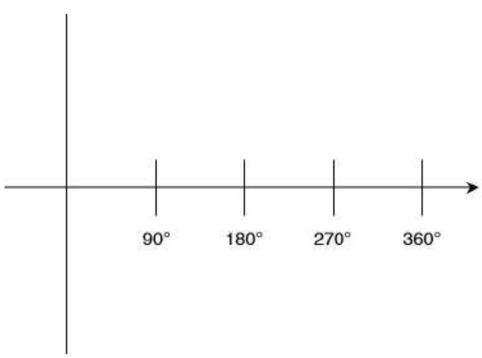
13. $y = 7\sin\theta$



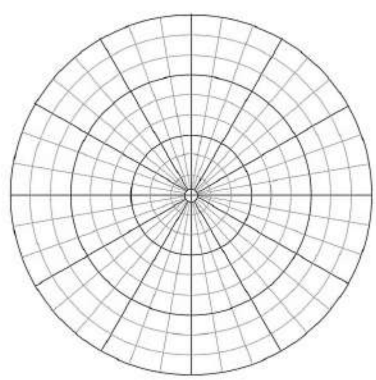

14. $r = 7\sin\theta$



15. $y = 4\sin(3\theta)$




16. $r = 4\sin(3\theta)$



## GO

Topic: Using the definition of logarithm to solve for $x$.

**Use the definition of $log\ x$ to find the value of $x$. Recall that $log\ x$ has a base of 10.**
(NO CALCULATORS)

17. $log\ x = 3$

18. $log\ x = -4$

19. $log\ x = 1$

20. $log\ x = 0$

21. $log\ x = \frac{1}{2}$

22. $log\ 10^7 = x$

23. $log\ 0.001 = x$

24. $log\ 1{,}000{,}000 = x$

25. $log\ 10^e = x$

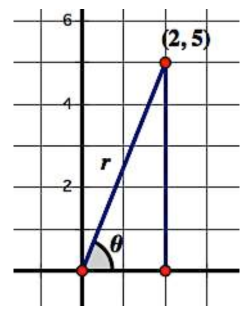
# 7.11H Complex Polar Forms

## *A Solidify Understanding Task*



Alyce and Veronica recall that they have learned how to represent complex numbers as points or vectors on a complex plane by letting the $x$-axis represent the real component of the complex number, and the $y$-axis representing the imaginary component. They are wondering if complex numbers can be related to Javier's work with the polar grid. From the internet, they learn that complex numbers can be expressed in polar form, and that much of the arithmetic of complex numbers can be simplified and enhanced when complex numbers are written in polar form. They are using the following problems to introduce Javier to the key ideas.

1. The point (2, 5) can be plotted on a rectangular grid. Find the polar coordinates for the same point. (Express the angle measure to the nearest tenth of a degree.)




2. In the complex plane, the point (2, 5) represents the complex number $2 + 5i$. In general, any point $(a, b)$ represents a corresponding complex number $a + bi$. How is the horizontal distance $a$ and the vertical distance $b$ represented in polar form? That is, how can we use $r$ and $\theta$ to describe the complex number $a + bi$?

**The arithmetic of complex numbers from a polar perspective**

**Multiplying complex numbers:**

Alyce and Veronica have learned that when complex numbers are written in complex form, $r \cdot \cos\theta + i \cdot r \cdot \sin\theta$, the product of two complex numbers is easy to find.

"You just multiply the *r's* and add the $\theta$'s," Veronica says excitedly.

Javier writes out in symbols what Veronica has claimed:

$$\text{If } a_1 + b_1 i = r_1\cos\theta_1 + ir_1\sin\theta_1 \text{ and } a_2 + b_2 i = r_2\cos\theta_2 + ir_2\sin\theta_2,$$

$$\begin{aligned}\text{then } (a_1 + b_1 i)(a_2 + b_2 i) &= (r_1\cos\theta_1 + ir_1\sin\theta_1)(r_2\cos\theta_2 + ir_2\sin\theta_2)\\ &= (r_1 \cdot r_2)\cos(\theta_1 + \theta_2) + i(r_1 \cdot r_2)\sin(\theta_1 + \theta_2)\\ &= a_3 + b_3 i\end{aligned}$$

Javier doesn't understand how Veronica's claim can be true, that is,

$(r_1\cos\theta_1 + ir_1\sin\theta_1)(r_2\cos\theta_2 + ir_2\sin\theta_2) = (r_1 \cdot r_2)\cos(\theta_1 + \theta_2) + i(r_1 \cdot r_2)\sin(\theta_1 + \theta_2)$

so he decides to try out her rule for a specific example.

3. Javier's experiment:

   a. Pick two complex numbers written in the form $a + bi$ and multiply them together algebraically, as you normally would.

   b. Rewrite both of the complex numbers in polar form.

   c. Multiply the polar forms of the two complex numbers together using Veronica's rule.

   d. Convert the product from polar form back to $a + bi$ form.

   e. Did you get the same result using Veronica's rule as you got in part a?

Javier is more convinced, but would like some proof that Veronica's rule will work all of the time, and not just for the few examples he tried. Veronica says, "As I recall, you have to use the sum and difference identities for sine and cosine to prove it." Javier decides to try to prove Veronica's rule.

4. Javier's proof:

   a. Multiply out $(r_1 cos\theta_1 + ir_1 sin\theta_1)(r_2 cos\theta_2 + ir_2 sin\theta_2)$ as the product of two binomials.

   b. Simplify the results using the sum and difference identities for sine and cosine that you wrote in the task *Double Identity*.

   c. Manipulate your final expression until it matches Veronica's claim.

**Dividing complex numbers:**

Alyce says, "We also learned that to divide complex numbers you divide the *r's* and subtract the *θ's*." This claim seems even more amazing to Javier. He decides to repeat the work he did with Veronica's claim.

5. Write out Alyce's claim symbolically.

6. Try out an experiment with Alyce's claim by selecting two complex numbers in the form $a + bi$ and divide them algebraically, as you normally would using the conjugate. Then write the complex numbers in polar form and divide them using Alyce's claim. Do you get the same result?

7. If your examples seem to support Alyce's claim, prove it using trig identities.

**Powers and roots of complex numbers:**

Javier has a new insight of his own as he thinks about Veronica's rule for multiplying complex numbers in polar form. "Since raising something to the $n^{th}$ power is just like repeated multiplication, we can write a rule for finding powers of a complex number written in polar form which will be much easier than multiplying out $(a + bi)^n$. Sweet!"

8. Finish Javier's rule for $(a + bi)^n = (r \cdot \cos\theta + i \cdot r\sin\theta)^{\ n} =$______________________________

Javier is wondering what happens to complex numbers as they are raised to higher and higher powers. He starts with the complex number $1 + \sqrt{3}i = 2\cos 60° + 2i\sin 60°$, since $r = 2$ and $\theta = 60°$ are both nice integer numbers.

9. Use Javier's rule to raise $1 + \sqrt{3}i$ to the following powers by using the polar form of this complex number, $2\cos 60° + 2i\sin 60°$, then convert the resulting complex number in polar form back to the form $a + bi$:

a. $\left(1 + \sqrt{3}i\right)^1 =$

b. $\left(1 + \sqrt{3}i\right)^2 =$

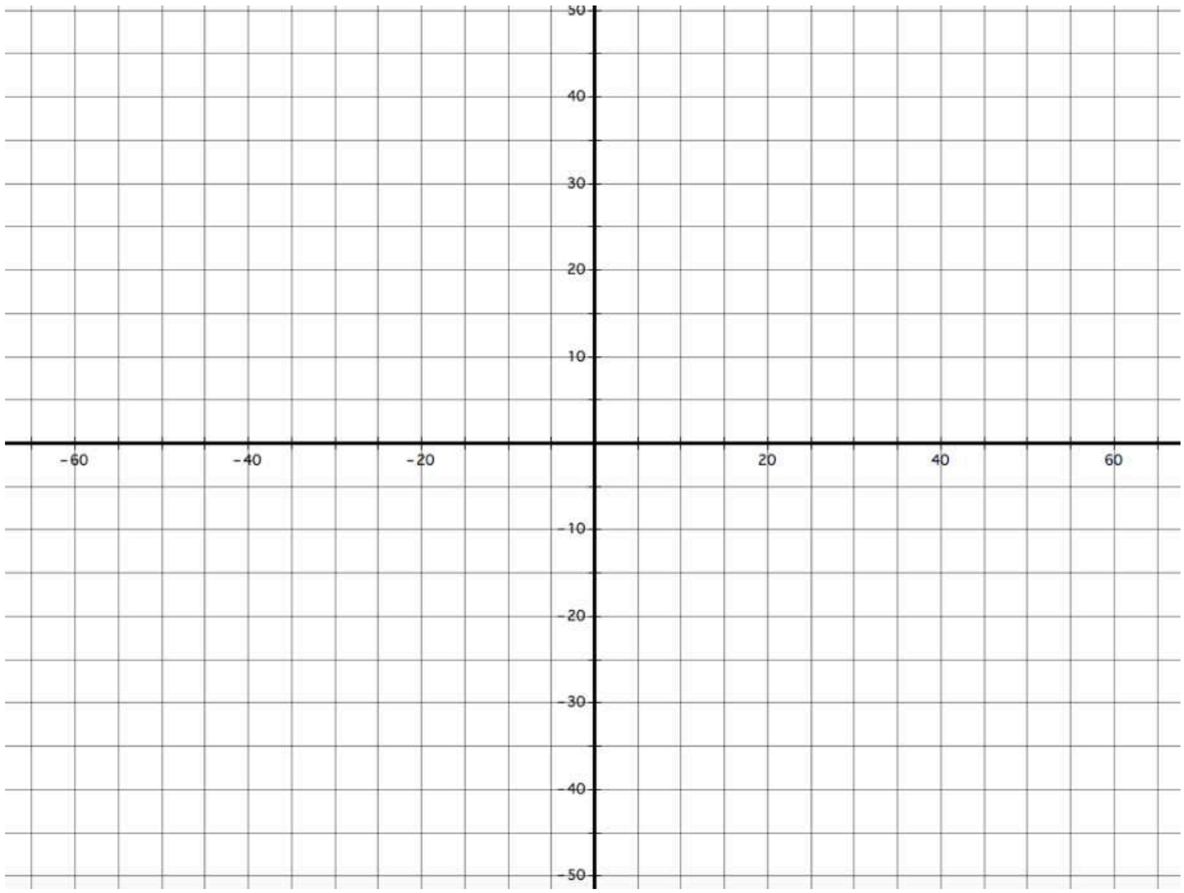
c. $\left(1 + \sqrt{3}i\right)^3 =$

d. $\left(1 + \sqrt{3}i\right)^4 =$

e. $\left(1 + \sqrt{3}i\right)^5 =$

f. $\left(1 + \sqrt{3}i\right)^6 =$

10. Plot each of the above complex numbers on the following complex plane. That is, treat the horizontal axis as a real number axis, and the vertical axis as an imaginary number axis. A complex number $a + bi$ is plotted as the point $(a, b)$. How does your understanding of exponential growth show up in the diagram? In the polar form of the powers?




Javier wonders if his rule can also be used to find roots of complex numbers. The rectangular form of the complex number you found in 9f, $64\cos 360° + 64i\sin 360°$, is $64 + 0i$. Javier knows that $\sqrt[3]{64} = 4$, since $4 \cdot 4 \cdot 4 = 64$. He decides to try to find the cube root of 64 using his rule applied to the polar form of the complex number $64 + 0i$.

11. Find the cube root of 64 using Javier's rule.

$$\sqrt[3]{64 + 0i} = (64\cos 360° + 64i\sin 360°)^{\frac{1}{3}} =$$

Verify that your answer is a cube root of 64 by multiplying it by itself three times:

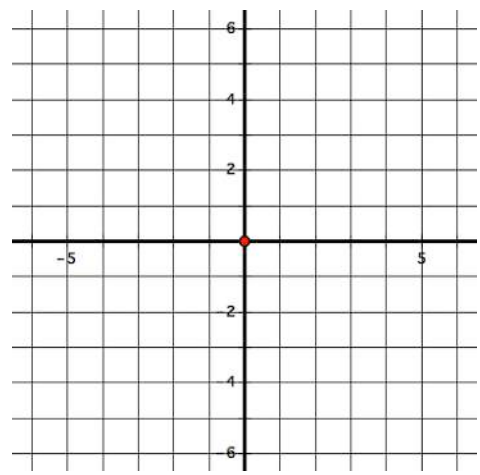
Javier is puzzled by the above results. He knows that $\sqrt[3]{64} = 4$, but this work with complex numbers suggests that $\sqrt[3]{64} = -2 + 2\sqrt{3}i$ is also a cube root of 64 if he considers complex numbers as possible cube roots. He wonders if he is missing any others. Javier decides to do some of his own searching on the internet, to see if he can find an answer to his question. Here are some of the results of his search:

Idea #1: If $x^n = a + bi$, we expect *n* complex roots of *a* + *bi.*

Idea #2: The *n* roots are $\dfrac{360°}{n}$ apart.

Idea #3: $(r\cos\theta + i \cdot r\sin\theta)^{\frac{1}{n}} = \left(r^{\frac{1}{n}} \cdot \cos\left(\frac{k\theta}{n}\right) + i \cdot r^{\frac{1}{n}} \cdot \sin\left(\frac{k\theta}{n}\right)\right) \quad for\ k = 1, 2, 3, \cdots, n$

12. Based on these ideas, what are the three cubes roots of 64?




Plot the cube roots of 64 on the complex plane:

READY, SET, GO! Name Period Date

## READY

Topic: Recalling the complex plane

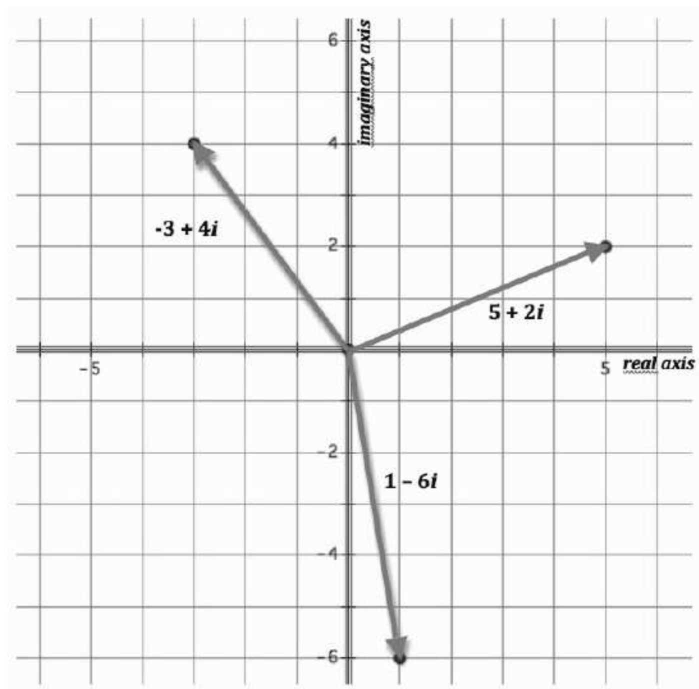
Recall from previous courses, that just as real numbers can be represented by points on the real number line, you can represent a complex number $\boldsymbol{z = a + bi}$ as the ordered pair $(a, b)$ in a coordinate plane called the **complex plane**. The horizontal axis is called the **real axis** and the vertical axis is called the **imaginary axis**. A complex number $a + bi$ can also be represented by a position vector with its tail located at the point $(0, 0)$ and its head located at the point $(a, b)$, as shown in the diagram. It will be useful to be able to move back and forth between both geometric representations of a complex number in the complex plane—sometimes representing the complex number as a single point, and sometimes as a vector.

**In the diagram, 3 complex numbers have been graphed as vectors. Rewrite each complex number as a point in the form (a, b).**

1. $-3 + 4i$ graphs as ( )
2. $5 + 2i$ graphs as ( )
3. $1 - 6i$ graphs as ( )

**On the diagram, graph the following complex numbers as vectors.**

4. $-5 - 3i$
5. $2 + 4i$
6. $-6 + i$
7. $2 - i$




Topic: Defining the modulus of a complex number

We can compare the relative magnitudes of complex numbers by determining how far they lie away from the origin in the complex plane. We refer to the magnitude of a complex number as its **modulus** and symbolize this length with the notation $|a + bi|$ where $|a + bi| = \sqrt{a^2 + b^2}$.

8. Find the modulus of each of the complex numbers in problems 1 - 7.

1) 2) 3) 4)

5) 6) 7)

## SET

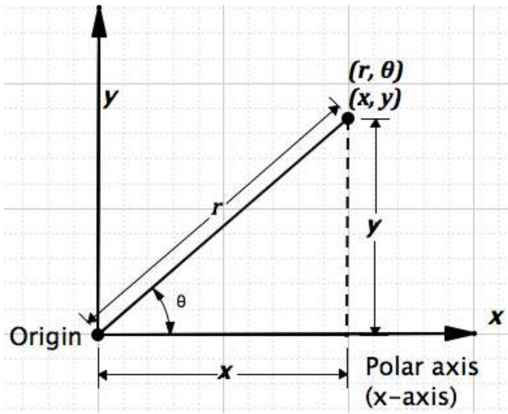
Topic: Connecting polar coordinates and rectangular coordinates

**Coordinate Conversion:**

The polar coordinates $(r, \theta)$ are related to the rectangular coordinates $(x, y)$ as follows:

$\boldsymbol{x = r\cos\theta}$ $\boldsymbol{y = r\sin\theta}$ $\boldsymbol{\tan\theta = \frac{y}{x}}$

$\boldsymbol{r^2 = x^2 + y^2}$




**Convert the points from polar to rectangular coordinates.**

9. $\left(4, \frac{\pi}{2}\right)$ 10. $\left(\sqrt{3}, \frac{5\pi}{6}\right)$ 11. $(2, \pi)$ 12. $\left(5\sqrt{2}, \frac{7\pi}{4}\right)$

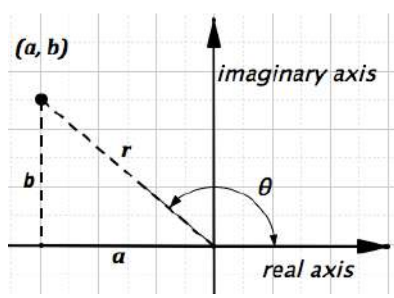
**Convert the points from rectangular to polar coordinates.**

13. $(-3, 3)$ 14. $(-6, 0)$ 15. $(1, \sqrt{3})$ 16. $(-4\sqrt{3}, -4)$

Topic: Writing the trigonometric form of a complex number

Consider the complex number $a + bi$. The angle $\theta$ is the angle measure from the positive real axis to the line segment connecting the origin and the point $(a, b)$.

$\boldsymbol{a = r\cos\theta \;\; and \;\; b = r\sin\theta, where\; r = \sqrt{a^2 + b^2}}.$

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By replacing $a$ and $b$, you have $\boldsymbol{a + bi = (r\cos\theta) + (r\sin\theta)i}$. Factor out the $r$ to obtain the **trigonometric form of a complex number.**

If $\boldsymbol{z = a + bi}$ then the trigonometric form is $\boldsymbol{z = r(\cos\theta + i\sin\theta)}$.

**Write the complex numbers in trigonometric form $\boldsymbol{z = r(\cos\theta + i\sin\theta)}$.**

17. $-3 - i$

18. $3 - 3i$

19. $-7 + 4i$

20. $\sqrt{3} + i$

**Write the complex number in standard form $\boldsymbol{a + bi}$.**

21. $3(\cos 120° + \sin 120°)$

22. $5(\cos 135° + i\sin 135°)$

23. $\sqrt{8}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$

24. $8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

## GO

Topic: Practicing operations on complex numbers

**Perform the indicated operation. Write your answers in standard form.**

25. $(4 + 7i) + (12 - 2i)$

26. $(11 - 8i) + (-4 - 3i)$

27. $(10 + 6i) - (16 - 3i)$

28. $(-7 - i) - (9 + i)$

29. $(1 + i)(4 - 2i)$

30. $(5 + 6i)(5 - 6i)$