mathematics vision project

Transforming Mathematics Education

SECONDARY MATH THREE

An Integrated Approach

Standard Teacher Notes

MODULE 7 HONORS

Trigonometric Functions, Equations & Identities

MATHEMATICSVISIONPROJECT.ORG

The Mathematics Vision Project

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7.1 High Noon and Sunset Shadows

A Develop Understanding Task

In this task we revisit the amusement park Ferris wheel that caused Carlos so much anxiety. Recall the following facts from previous tasks:

- The Ferris wheel has a radius of 25 feet
- The center of the Ferris wheel is 30 feet above the ground
- The Ferris wheel makes one complete rotation counterclockwise every 20 seconds

The amusement park Ferris wheel is located next to a high-rise office complex. At sunset, the moving carts cast a shadow on the exterior wall of the high-rise building. As the Ferris wheel turns, you can watch the shadow of a rider rise and fall along the surface of the building. In fact, you know an equation that would describe the motion of this "sunset shadow."

1. Write the equation of this "sunset shadow."

At noon, when the sun is directly overhead, a rider casts a shadow that moves left and right along the ground as the Ferris wheel turns. In fact, you know an equation that would describe the motion of this "high noon shadow."

2. Write the equation of this "high noon shadow."

3. Based on your previous work, you probably wrote these equations in terms of the angle of rotation being measured in degrees. Revise you equations so the angle of rotation is measured in radians.

    a. The "sunset shadow" equation in terms of radians:

    b. The "high noon shadow" equation in terms of radians:

4. In the equations you wrote in question 3, where on the Ferris wheel was the rider located at time $t = 0$? (We will refer to the position as the rider's *initial position* on the wheel.)

5. Revise your equations from question 3 so that the rider's initial position at $t = 0$ is at the top of the wheel.

    a. The "sunset shadow" equation, initial position at the top of the wheel:

    b. The "high noon shadow" equation, initial position at the top of the wheel:

6. Revise your equations from question 3 so that the rider's initial position at $t = 0$ is at the bottom of the wheel.

    a. The "sunset shadow" equation, initial position at the bottom of the wheel:

    b. The "high noon shadow" equation, initial position at the bottom of the wheel:

7. Revise your equations from question 3 so that the rider's initial position at $t = 0$ is at the point farthest to the left of the wheel.

   a. The "sunset shadow" equation, initial position at the point farthest to the left of the wheel:

   b. The "high noon shadow" equation, initial position at the point farthest to the left of the wheel:

8. Revise your equations from question 3 so that the rider's initial position at $t = 0$ is halfway between the farthest point to the right on the wheel and the top of the wheel.

   a. The "sunset shadow" equation, initial position halfway between the farthest point to the right on the wheel and the top of the wheel:

   b. The "high noon shadow" equation, initial position halfway between the farthest point to the right on the wheel and the top of the wheel:

9. Revise your equations from question 3 so that the wheel rotates twice as fast.

   a. The "sunset shadow" equation for the wheel rotating twice as fast:

   b. The "high noon shadow" equation for the wheel rotating twice as fast:

10. Revise your equations from question 3 so that the radius of the wheel is twice as large and the center of the wheel is twice as high.

    a. The "sunset shadow" equation for a radius twice as large and the center twice as high:

    b. The "high noon shadow" equation for a radius twice as large and the center twice as high:

11. Carlos wrote his "sunset equation" for the height of the rider in question #5 as $h(t) = 50\sin\left(\frac{\pi}{10}t + \frac{\pi}{2}\right) + 30$. Clarita wrote her equation for the same problem as $h(t) = 50\sin\left(\frac{\pi}{10}(t+5)\right) + 30$.

    a. Are both of these equations equivalent? How do you know?

    b. Carlos says his equation represents starting the rider at an initial position at the top of the wheel. What does Clarita's equation represent?

7.1 High Noon and Sunset Shadows – Teacher Notes

A Develop Understanding Task

Purpose: The purpose of this task is to develop strategies for transforming the Ferris wheel functions so that the function and graphs represent different initial starting positions for the rider. Students have already considered vertical translations by moving the center of the Ferris wheel up or down, resulting in the midline of the graph being translated vertically. They have also considered horizontal and vertical dilations of the graph by changing the radius of the Ferris wheel or the speed of rotation, resulting in varying the amplitude or the period of the graph. In this task, horizontal translations of the graph are considered. Students may also note that sine and cosine graphs are interchangeable, as long as the graph is shifted horizontally by an appropriate amount.

Core Standards Focus:

F.TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

F.BF.3 Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

Related Standards: F.BF.1c

Standards for Mathematical Practice:

SMP 4 – Model with mathematics

SMP 5 – Use appropriate tools strategically

Vocabulary: This task varies the *initial position* of the rider on the Ferris wheel.

The Teaching Cycle:

Launch (Whole Class):

Model the scenario given in this task by using a strong flashlight to represent the sun. Have a student hold a pencil so it is parallel to the ground and move it in a continual circular path to model the rider on the Ferris wheel. Note the shadow of the pencil as it moves up and down the wall when the flashlight is pointed horizontally towards the pencil (the "sunset shadow") or the shadow of the pencil as it moves back and forth across the ground when the flashlight is held above the pencil and pointed towards the ground (the "high noon shadow").

After modeling the context, assign students to answer questions 1-3 on the first page of the task. Remind students that they have written equations for the vertical and horizontal motion of the rider on this Ferris wheel in terms of degrees on previous tasks (see 6.2 and 6.5). Question 3 asks students to revise these equations in terms of radians. (Note that we will use *y(t)* to represent the height of the rider—the motion of the sunset shadow—and *x(t)* to represent the horizontal position of the rider—the motion of the high noon shadow. You may want to suggest this notation to your students, also.

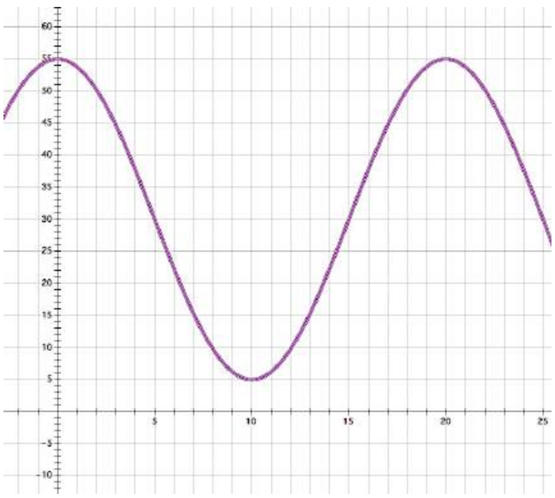
$$y(t) = 25\sin\left(\frac{\pi}{10}t\right) + 30$$
$$x(t) = 25\cos\left(\frac{\pi}{10}t\right)$$

Once students have agreed on the equations for the motion of the high noon and sunset shadows, and can explain why $b = \frac{\pi}{10}$ in the expression $a\sin(bt) + d$, assign them to work on the remainder of the task where the *initial* or *starting position* of the rider will be located at different points around the wheel.

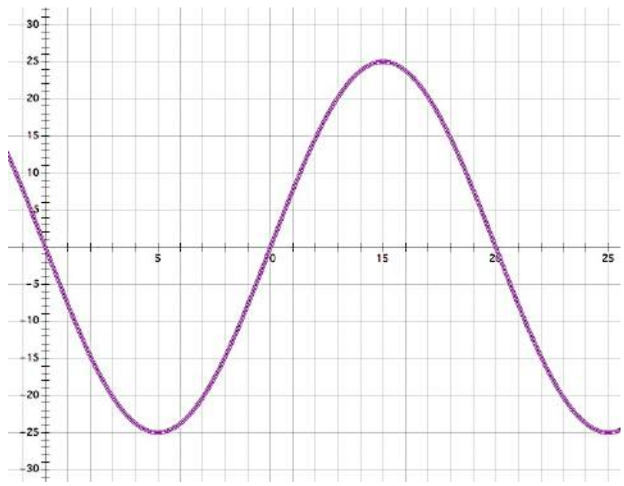
Explore (Small Group):

If students are having difficulty getting started on this work, suggest that they sketch the graphs of the motion of the high noon and sunset shadows such that at time $t = 0$ the rider is at the initial position given in the scenario. This graphical representation of the rider should help them think about how they might modify their equations from question 3 to produce these new graphs. Students might use technology to test out their proposed equations to see if they match the graphs they have sketched by hand. For example, for question 5 (starting position of the rider at the top of the wheel), students should sketch the following graphs:

Sunset Shadow



High Noon Shadow



Students should notice that the graph representing the height of the rider—the motion of the sunset shadow—looks like a cosine graph, and that the graph representing the horizontal position of the rider—the motion of the high noon shadow—looks like a sine graph that has been reflected over the x-axis. This should lead to the following proposals for the functions:

$$y(t) = 25\cos\left(\frac{\pi}{10}t\right) + 30$$

$$x(t) = -25\sin\left(\frac{\pi}{10}t\right)$$

Accept these as correct equations, and then challenge students to try to find alternative equations so that we can continue to use a sine function to represent the vertical motion of the rider and a cosine function to represent the horizontal motion. Suggest that they might do so by modifying the expression inside the function, thereby translating the sine or cosine function horizontally. Let students guess and check different values for c in the expression $a\sin(bt + c) + d$ if that is their preferred strategy, but ultimately students should be able to explain why their choice for c makes sense. Press for explanations such as, "At time $t = 0$ we want the function to use the value of the

$\sin\frac{\pi}{2}$, so we will start the rider at the highest position on the graph."

It may be helpful to hold a short discussion regarding students' results on question 5 before having students work on the remainder of the task. Students should be able to explain why the following equations also work for the graphs in question 5:

$$y(t) = 25\sin\left(\frac{\pi}{10}t + \frac{\pi}{2}\right) + 30$$
$$x(t) = 25\cos\left(\frac{\pi}{10}t + \frac{\pi}{2}\right)$$

Ask students to continue to find ways to use a sine function to represent the vertical motion of the rider and a cosine function to represent the horizontal motion of the rider as they work on the remainder of the problems in the task. This will keep the task focused on the horizontal transformations of the graphs. If some students finish the task more quickly than others, challenge them to go back and write other equations for each situation that might involve a different trigonometric function or a different horizontal transformation.

Discuss (Whole Class):

Post these generic forms for describing the motion of the high noon and sunset shadows:

The sunset shadow: $y(t) = a\sin(bt + c) + d$

The high noon shadow: $x(t) = a\cos(bt + c) + d$

Point out to students that the parameter c translates the sine or cosine graph horizontally and is sometimes referred to as a *phase shift* in scientific applications.

Select students who can explain their choices of values for c to present problems 6-8. After an equation has been presented and explained, it may be helpful to have other students calculate a few specific values of the function, such as at $t = 0$, $t = 5$, $t = 10$ or $t = 15$ seconds, to verify that the function works at these specific instances in time.

Given time, you might want to have students present some of the alternative equations they found to describe these graphs, pointing out that sine and cosine functions can be used interchangeably, as long as appropriate phase shifts are considered.

Aligned Ready, Set, Go: Trigonometric Functions, Equations and Identities 7.1

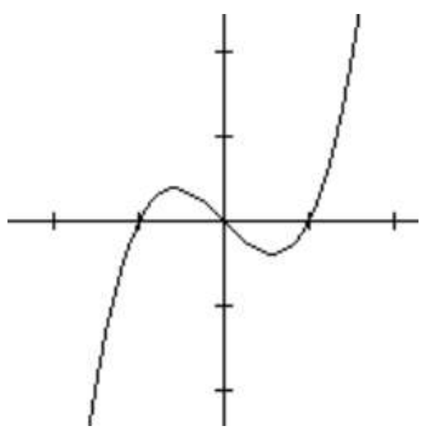
READY, SET, GO! Name Period Date

READY

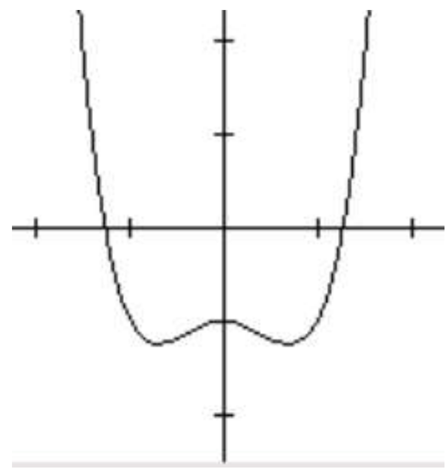
Topic: Recalling invertible functions and even and odd functions

Indicate which of the following functions have an inverse that is a function. If the function has an inverse, sketch it in. (Remember, the inverse will reflect across the y = x line. Sketch that in, too.) Finally, label each one as *even, odd*, or *neither*. Recall that an *even* function is symmetric with the y-axis, while an *odd* function is symmetric with respect to the origin.

1,



2.



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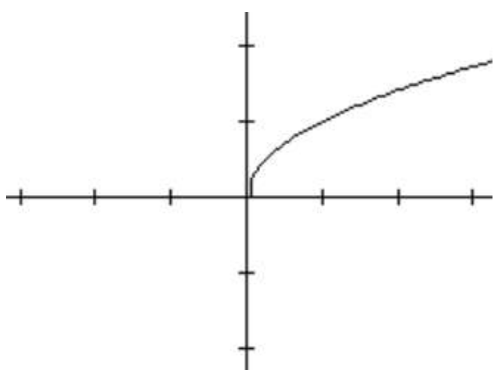

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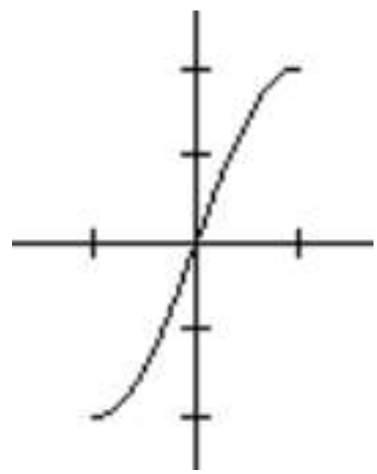
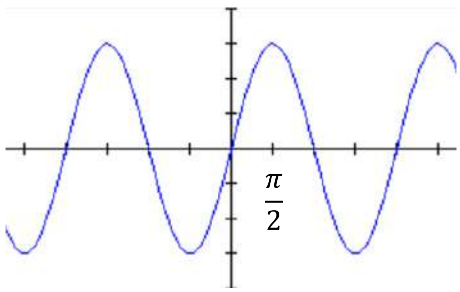


SET

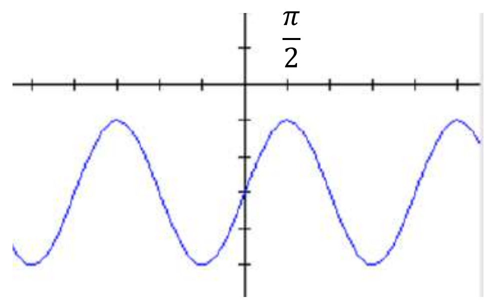
Topic: Connecting transformed trig graphs with their equations

State the period, amplitude, vertical shift, and phase shift of the function shown in the graph. Then write the equation. Use the same trigonometric function as the one that is given.

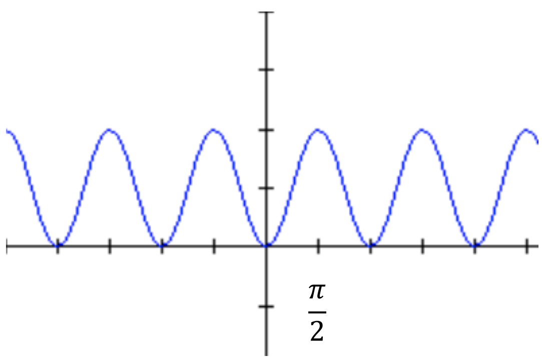
5. $y = sin\,x$



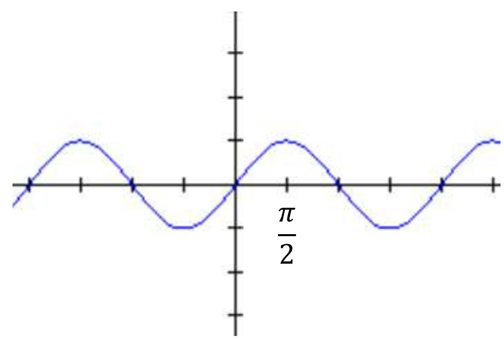
6. $y = sin\,x$



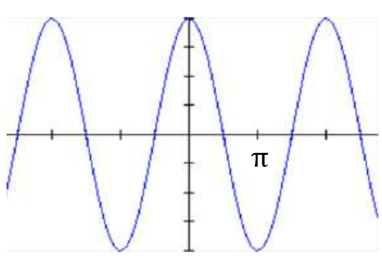
7. $y = cos\,x$



8. $y = cos\,x$



9. $y = sin\,x$



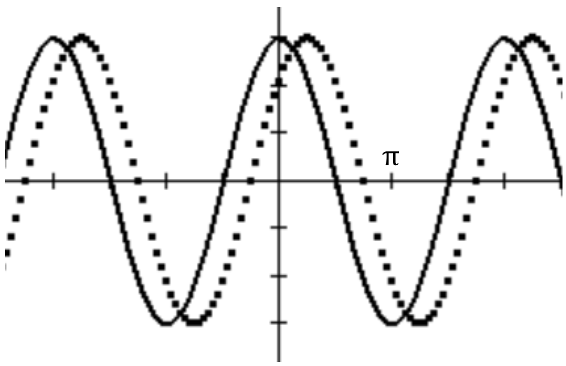
10. The cofunction identity states that $\sin\theta = \cos(90° - \theta)$ *and* $\sin(\theta - 90°) = \cos\theta$. How does this identity relate to the graph in #9?

Explain where you would see this identity in a right triangle.

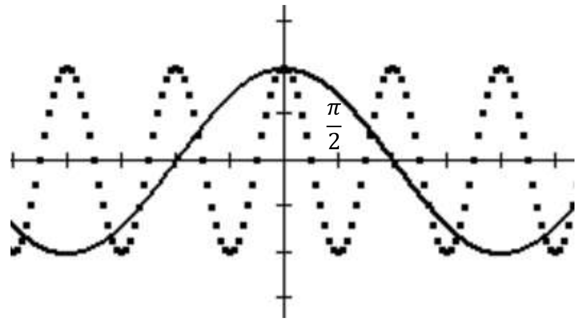
Describe the relationships between the graphs of *f (x)* - solid and *g (x)* - dotted.

Then write their equations.

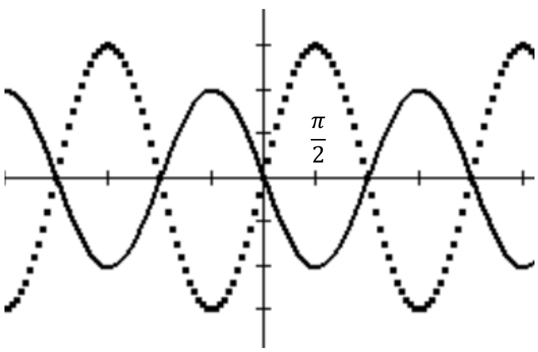
11.



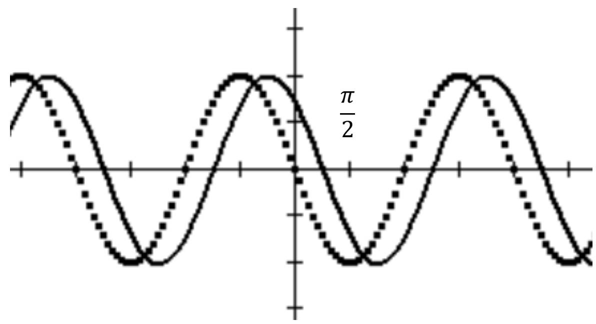
12.



13. This graph could be interpreted as a shift or a reflection. Write the equations both ways.

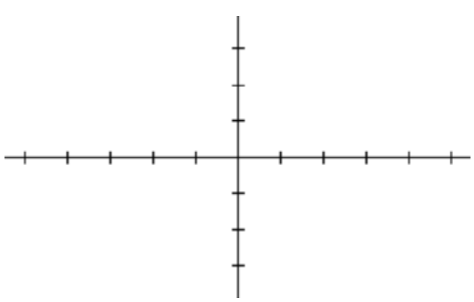


14.

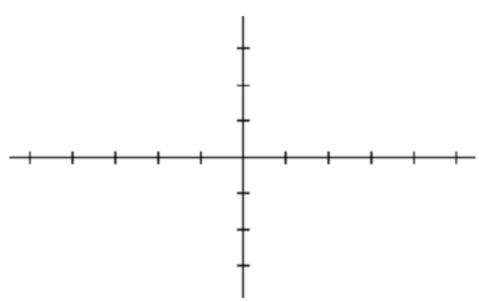


Sketch the graph of the function.
(Include 2 full periods. Label the scale of your horizontal axis.)

15. $y = 3\sin\left(x - \frac{\pi}{2}\right)$



16. $y = -2\cos(x + \pi)$



GO

Topic: Finding angles of rotation for the same trig ratio

Name two values for θ (angles of rotation) that have the given trig ratio. $0 < \theta \leq 2\pi$.

17. $\sin\theta = \frac{\sqrt{2}}{2}$

18. $\cos\theta = \frac{\sqrt{2}}{2}$

19. $\cos\theta = -\frac{1}{2}$

20. $\sin\theta = 0$

21. $\sin\theta = -\frac{\sqrt{3}}{2}$

22. $\cos\theta = -\frac{\sqrt{3}}{2}$

23. For which angles of rotation does $\sin\theta = \cos\theta$?

7.2 High Tide

A Solidify Understanding Task

Perhaps you have built an elaborate sand castle at the beach only to have it get swept away by the in-coming tide.

Spring break is next week and you are planning another trip to the beach. This time you decide to pay attention to the tides so that you can keep track of how much time you have to build and admire your sand castle.

You have a friend who is in calculus who will be going on spring break with you. You give your friend some data from the almanac about high tides along the ocean, as well as a contour map of the beach you intend to visit, and ask her to come up with an equation for the water level on the beach on the day of your trip. According to your friend's analysis, the water level on the beach will fit this equation:

$$f(t) = 20\,sin\left(\frac{\pi}{6}t\right)$$

In this equation, $f(t)$ represents how far the waterline is above or below its average position. The distance is measured in feet, and t represents the elapsed time (in hours) since midnight.

1. What is the highest up the beach (compared to its average position) that the waterline will be during the day? (This is called *high tide*.) What is the lowest that the waterline will be during the day? (This is called *low tide*.)

2. Suppose you plan to build your castle right on the average waterline just as the water has moved below that line. How much time will you have to build your castle before the incoming tide destroys your work?

3. Suppose you want to build your castle 10 feet below the average waterline to take advantage of the damp sand. What is the maximum amount of time you will have to make your castle? How can you convince your friend that your answer is correct?

4. Suppose you want to build your castle 15 feet above the average waterline to give you more time to admire your work. What is the maximum amount of time you will have to make your castle? How can you convince your friend that your answer is correct?

5. You may have answered the previous questions using a graph of the tide function. Is there a way you could use algebra and the inverse sine function to answer these questions. If so, show your work.

    a. Algebraic work for question 3:

b. Algebraic work for question 4:

6. Suppose you decide you only need two hours to build and admire your castle. What is the lowest point on the beach where you can build it? How can you convince your friend that your answer is correct?

7.2 High Tide – Teacher Notes

A Solidify Understanding Task

Purpose: In this task students solidify their understanding of using trigonometric functions to model periodic behavior by applying trigonometry to a context that is periodic (high and low tides), but not circular motion. They learn how to interpret amplitude and period in terms of this new context. They also consider using inverse trigonometric functions to answer questions about the time at which the tide reaches various heights. Students answer these inverse questions both graphically and algebraically.

Core Standards Focus:

F.TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

F.BF.4 Find inverse functions.

a. Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse.

c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.

Standards for Mathematical Practice:

SMP 4 – Model with mathematics

The Teaching Cycle:

Launch (Whole Class):

Introduce the context of tides to students and have them answer questions 1 and 2 individually, and then discuss their answers as a class. Verify that students are able to interpret the parameters of this equation in terms of this new context. They should be able to explain that the 20 in the equation tells them that the tide will range 20 feet above or below average tide, and that the $^{\pi}/_{6}$

tells them that the period of the tide function (one complete cycle of high and low tides) is 12 hours, which can be found by calculating $\frac{2\pi}{\pi/6}$. If students are having difficulty relating a trigonometric function to a non-circular context, ask then how they would answer these questions if the equation was referring to a Ferris wheel. Point out that even though the height of the tides has nothing to do with angles of rotation, we can "borrow" the periodic behavior of circular trigonometric functions to describe other periodic contexts such as tides, vibrating strings, or the cycle of average temperatures over the course of a year.

Following this introduction, have students work on questions 3-5, and question 6 if they have time.

Explore (Small Group):

It is anticipated that students will answer questions 3 and 4 by referring to the graph of the situation, perhaps by plotting a horizontal line at y = -10 (for question 3) or y = 15 (for question 4) and then using the features of the calculator to determine the points of intersection of these lines with the trigonometric graph. This strategy is an important one for all students to consider as it develops conceptual understanding for the problem, so encourage and support it. Select students to present during the discuss stage of the lesson who can describe their work using this strategy. Also watch for students who use tables to answer questions 3 and 4 and select students to present this strategy also, if it is present in the work of the students. Note that since the tide goes through two complete cycles during a 24-hour day, there are actually two intervals of time when the sand castle building criteria can be satisfied for questions 3 and 4. Make sure students are answering the questions in terms of the time of the day—morning, afternoon or evening—when the sand castle could be built.

Students should be familiar with using inverse functions, and using inverse trigonometric functions to find missing angles in right triangles. They probably will not have considered the issues that arise when inverse trigonometric functions are applied to periodic functions. Allow these issues to surface naturally as students work on their algebraic approaches to answering questions 3 and 4 (see question 5). Listen for how students deal with these issues and select students to discuss these issues and how they resolved them during the whole class discussion. Referring to the graphs of the situations will help make these issues more clear. For example, students might perform the following algebra for question 3:

$$-10 = 20\sin\left(\frac{\pi}{6}t\right)$$
$$\frac{-10}{20}=\sin\left(\frac{\pi}{6}t\right)$$
$$-\frac{1}{2} = \sin\left(\frac{\pi}{6}t\right)$$
$$\sin^{-1}\left(-\frac{1}{2}\right) = \sin^{-1}\left(\sin\left(\frac{\pi}{6}t\right)\right)$$
$$\sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{6}t$$

At this point the question might arise, "What is the inverse sine of -1/2?" That is, "What is the angle whose sine is -1/2?" If we examine the unit circle we observe that the sine function has a value of -1/2 at the principal values of $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. Using these values to solve for t we find we can build our sand castle between 7:00 and 11:00 a.m. in the morning. However, the graph suggests another interval of time between 7:00 and 11:00 p.m. (or 19:00-23:00 military time) as a possible building time if we want to stay on the beach late in the evening. This interval works because the sine equals -1/2 at $\frac{19\pi}{6}$ and at $\frac{23\pi}{6}$ also. In fact, there are an infinite number of angles for which the sine of the angle is -1/2, but only these four angles make sense in our context of spending a day at the beach. If we turn to the calculator we get $\sin^{-1}\left(-\frac{1}{2}\right) = -0.5236\ or\ -\frac{\pi}{6}$, which really doesn't fit our context at all, since it would mean t = -1, or one hour before midnight of the day we plan to visit the beach. Allow students to wrestle with these ideas until they arrive at similar explanations in terms of the context, as given above. [Wait until the whole class discussion to point out the dilemma that the inverse sine function can only have one output, and it is by convention that mathematicians have restricted output values for the inverse sine function to the interval $\frac{\pi}{2} \le \sin^{-1}(x) \le \frac{\pi}{2}$. Note that the calculator output was in this interval.]

The algebra for question 4 follows the same reasoning as above, however there are no easily identifiable angles on the unit circle for the question, "What angle has a sine value of $^3/_4$?" Consequently, we are left solving this equation using a calculated value for $\sin^{-1}\left(\frac{3}{4}\right) = 0.848$. Solving the equation for t using this value yields t = 1.62, but we can't start building our sand castle

1.62 hours after midnight (approximately 1:37 a.m.), since the level of the tide is rising at that time, not receding. Students will have to reason through how they might use this value to determine the interval of time when they might build their castle. Examining a unit circle diagram would suggest that the sine function also has a value of $^3/_4$ at an angle equal to π – 0.848 or an angle measuring 2.2936 radians. Solving the equation for t using this value yields t = 4.38 hours after midnight, or approximately 4:23 a.m. as the time when we can start building our sand castle. To find the time when the rising tide will wash away our sand castle we will need to find the next angle where the sine function equals $^3/_4$. This occurs at 2π +0.848 or an angle measuring 7.1312 radians. Solving the equation for t using this value yields t = 13.62 hours after midnight or 1:37 p.m. as the end of the interval in which we can admire our work. (We could also build our sand castle between 4:23 p.m. and 1:37 a.m. the next morning.)

Discuss (Whole Class):

The discussion needs to focus on the ideas and strategies for answering inverse trigonometric questions, as highlighted by the issues addressed above. Discuss the convention for restricting the output of the inverse sine <u>function</u> to the interval $\frac{\pi}{2} \leq \sin^{-1}(x) \leq \frac{\pi}{2}$. This allows the domain of the inverse sine function to cover all possible values of x, $-1 \leq x \leq 1$, and yet yield a unique output. Consequently, the answer obtained using a calculator may not be the answer that fits the context. The unit circle and the graph of the situation serve as aids for figuring out how to interpret calculated values for the inverse sine so that results fit the scenario. Use student work and student thinking about questions 3 and 4 to help make these points. Start with graphs and/or tables to first find the answers to questions 3 and 4, and then pursue the algebraic ideas outlined above.

If there is time, discuss question 6 using tables and graphs to make sense of the scenario. This is not an inverse trigonometric question, since we need only evaluate the given function at particular times to find the lowest point on the beach where we can build the castle. However, it may be difficult to determine the times at which we should evaluate the function without the use of a table or a graph.

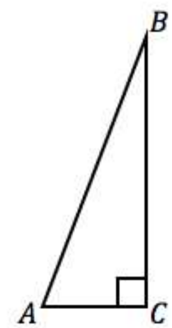
Aligned Ready, Set, Go: Trigonometric Functions, Equations and Identities 7.2

READY, SET, GO! Name Period Date

READY

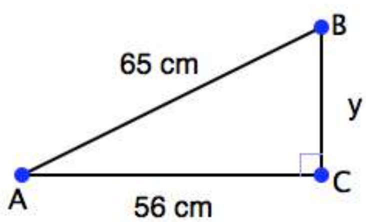
Topic: Calculating tangent in right angle trigonometry

Recall that the right triangle definition of the tangent ratio is:

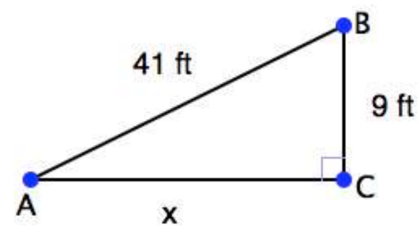


$$\tan A = \frac{\textit{length of side opposite angle } A}{\textit{length of side adjacent to angle } A}$$

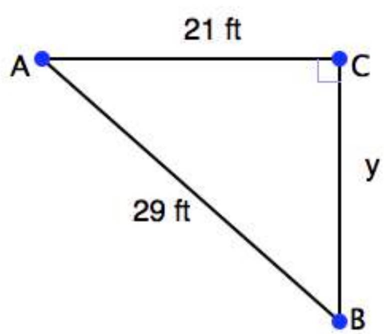
1. Find $\tan A$ *and* $\tan B$.



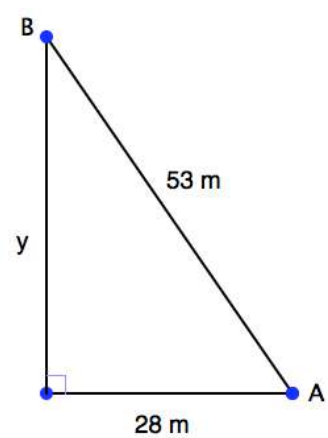
2. Find $\tan A$ *and* $\tan B$.



3. Find $\tan A$ *and* $\tan B$.



4. Find $\tan A$ *and* $\tan B$.



SET

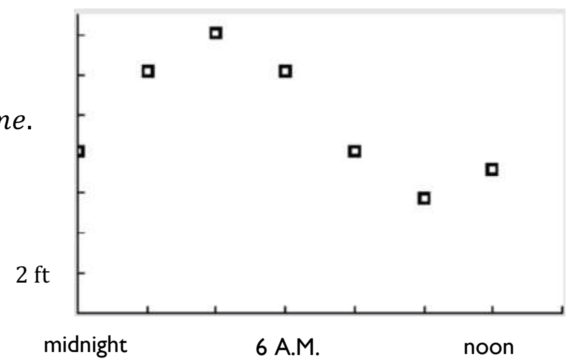
Topic: Mathematical modeling using sine and cosine functions

Many real-life situations such as sound waves, weather patterns, and electrical currents can be modeled by sine and cosine functions. The table below shows the depth of water (in feet) at the end of a wharf as it varies with the tides at various times during the morning.

t (time)	midnight	2 A.M.	4 A.M.	6 A.M.	8 A.M.	10 A.M.	noon
d (depth)	8.16	12.16	14.08	12.16	8.16	5.76	7.26

We can use a trigonometric function to model the data. Suppose you choose cosine.
$y = A\cos(bt - c) + d$, *where y is depth at any time.*

The *amplitude* will be the distance from the average of the highest and lowest values. This will be the average depth (d).



5. Sketch the line that shows the average depth.

6. Find the amplitude. $A = \frac{1}{2}(high - low)$

7. Find the period. $p = 2|low\ time - high\ time|$. Since a normal period for sine is 2π. The new period for our model will be $\frac{2\pi}{p}$ $so\ b = \frac{2\pi}{p}$. (Use the *p* you calculated, divide and turn it into a decimal.)

8. High tide occurred 4 hours after midnight. The formula for the displacement is $4 = \frac{c}{b}$. Use *b* and solve for *c.*

9. Now that you have your values for *A, b, c*, and *d*, put them into an equation.
$y = A\cos(bt - c) + d$

10. Use your model to calculate the depth at 9 A.M. and 3 P.M.

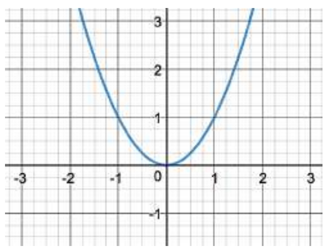
11. A boat needs at least 10 feet of water to dock at the wharf. During what interval of time in the afternoon can it safely dock?

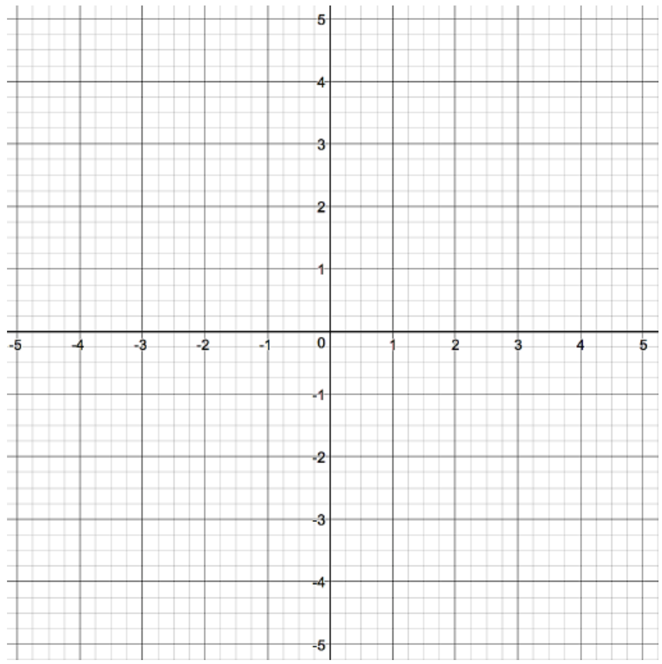
GO

Topic: Connecting transformations on functions

The equation and graph of a parent function is given. For each transformation, describe the change on the graph of the parent function. Then graph the functions on the same grid.

12. $f(x) = x^2$



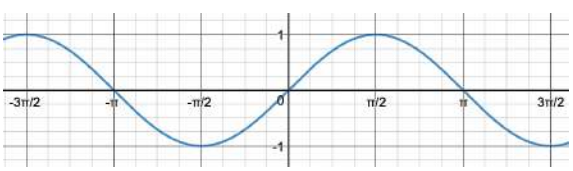


a. $f(x) = x^2 - 3$
Description:

b. $f(x) = (x-3)^2 - 4$
Description:

c. $f(x) = 2(x-3)^2 - 4$
Description:

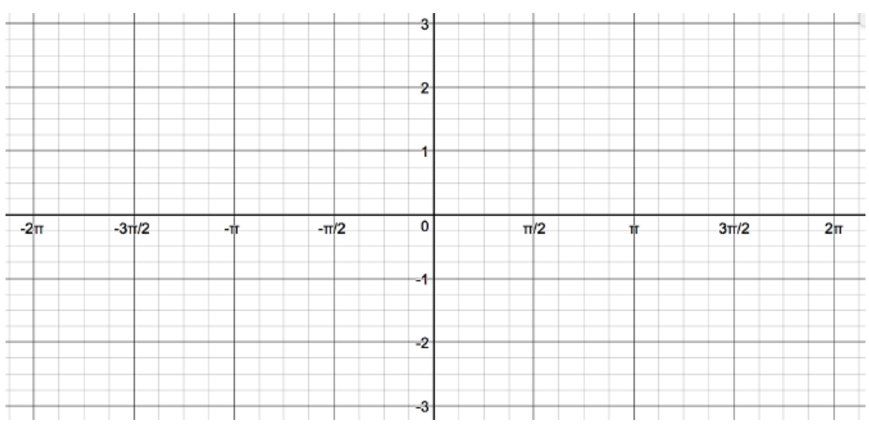
13. $g(x) = sinx$



a. $g(x) = (sinx) + 2$
Description:

b. $g(x) = sin\left(x + \frac{\pi}{2}\right) - 1$
Description:

c. $g(x) = 2sin\left(x + \frac{\pi}{2}\right) - 1$
Description:

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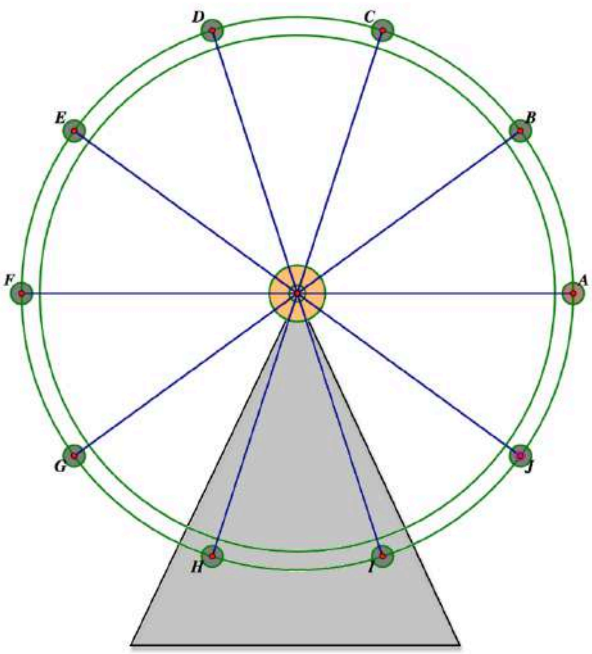
7.3 Getting on the Right Wavelength

A Practice Understanding Task

The Ferris wheel in the following diagram has a radius of 40 feet, its center is 50 feet from the ground, and it makes one revolution counterclockwise every 18 seconds.



1. Write the equation of the height of the rider at any time t, if at $t = 0$ the rider is at position A (Use radians to measure the angle of rotation).

2. At what time(s) is the rider 70 feet above the ground? Show the details of how you answered this question.

3. If you used a sine function in question 1, revise your equation to model the same motion with a cosine function. If you used a cosine function, revise your equation to model the motion with a sine function.

4. Write the equation of the height of the rider at any time t, if at t = 0 the rider is at position D (Use radians to measure the angle of rotation).

5. For the equation you wrote in question 4, at what time(s) is the rider 80 feet above the ground? Show or explain the details of how you answered this question.

6. Choose any other starting position and write the equation of the height of the rider at any time t, if at t = 0 the rider is at the position you chose. (Use radians to measure the angle of rotation). Also change other features of the Ferris wheel, such as the height of the center, the radius, the direction of rotation and/or the length of time for a single rotation. (Record your equation and description of your Ferris wheel here.)

7. Trade the equation you wrote in question 7 with a partner and see if he or she can determine the essential features of your Ferris wheel: height of center, radius, period of revolution, direction of revolution, starting position of the rider. Resolve any issues where you and your partner have differences in your descriptions of the Ferris wheel modeled by your equation.

7.3 Getting on the Right Wavelength – Teacher Notes

A Practice Understanding Task

Purpose: The purpose of this task is to practice the concepts and strategies associated with inverses and transformations of trigonometric functions in a modeling context. Students should use graphs produced with technology to verify their work.

Core Standards Focus:

F.TF.5 Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.★

F.BF.3 Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

F.BF.4 Find inverse functions.

a. Solve an equation of the form f(x) = c for a simple function f that has an inverse and write an expression for the inverse.

c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.

Related Standards: F.BF.1c

Standards for Mathematical Practice:

SMP 2 – Reason abstractly and quantitatively

Vocabulary: For trigonometric functions, a horizontal transformation of a graph is often referred to as a *phase shift*.

The Teaching Cycle:

Launch (Whole Class):

Tell students that they are going to be using all of the trigonometric ideas they have learned in this module and the previous module as they work on this task. Encourage them to use a calculator, set in radian mode, and the unit circle to help them reason through each of the questions. Inform them, however, that their answers to each question should include algebraic work and representations, in addition to answers obtained from a graphical analysis. Students should be able to explain the meaning of each of the parameters in their functions and how they used their functions to answer each question.

Explore (Small Group):

Listen for how students are reasoning through each problem and what tools they draw upon—the context, the Ferris wheel image, the graph, a table, the unit circle, or algebra—to answer each question. For example, once students have written the equation for the height of the rider on the Ferris wheel in question 1, they could use a graph to answer question 2. Acknowledge this approach as appropriate and correct, but also encourage them to think through the algebra of using an inverse sine function and the unit circle to answer the same question. In question 3 students could do some guess and check work to get a cosine graph to match the sine graph from question 1, but press them to explain how they could get the same parameter for the horizontal transformation by reasoning about the period and phase shift of the graph.

Questions 4-5 revisit the same work as questions 1-3, but with a greater level of challenge.

Questions 6-7 provide opportunities for students to create their own scenarios, and may surface additional issues to be resolved or they may highlight students' misconceptions.

Discuss (Whole Class):

Focus the discussion on how students used different tools—the context, the Ferris wheel image, the graph, a table, or algebra—to reason through each question. You may want to have more than one student presentation for each question to highlight different strategies. Connect each way of thinking to the algebraic representations for the problem. For example, on question 2 a student might note that a height of 70 feet is halfway between the center and the top of the Ferris wheel and might also relate this to the angles $^{\pi}/_{6}$ and $^{5\pi}/_{6}$ where the y-coordinate is ½ on the unit circle. Consequently, the rider will have completed $^{1}/_{12}$ (or $^{5}/_{12}$) of a revolution, which will take 1.5 (or 7.5) seconds. This type of reasoning can be related to corresponding work in the algebraic solution:

$$70 = 40\sin\left(\frac{\pi}{9}t\right) + 50$$
$$20 = 40\sin\left(\frac{\pi}{9}t\right)$$
$$\frac{1}{2} = \sin\left(\frac{\pi}{9}t\right)$$
$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{9}t$$
$$\frac{\pi}{6} = \frac{\pi}{9}t$$
$$t = \frac{3}{2}\, seconds$$

Reasoning from the diagram or from a graph we can see additional solutions to this question occur at 7. 5 seconds, 19.5 seconds, 25.5 seconds, etc. These times can be listed in the form 1.5 +18n seconds and 7.5 + 18n seconds, where n represents the set of natural numbers (or an appropriate subset, depending on the duration of the ride).

When starting the rider at a different position on the wheel at time t = 0, we need to shift the graph of the height of the rider left or right to correspond with this new starting position. We can reason about this shift in two ways: adding an angle to the angle represented by bt in the equation $h(t) = a\sin(bt + c) + d$, or by adjusting the time when the rider is at a particular starting position by adding a few additional seconds to the time in the equation $h(t) = a\sin(b(t + c)) + d$. If we add to the angle, or to the time, the graph shifts left so that this new starting position occurs at time t = 0.

That is, we are starting at a position that is a few radians or a few seconds into the standard ride. Likewise, if we subtract from the angle, or from the time, the graph shifts right so that this new starting position occurs at time $t = 0$. That is, we are starting at a position that occurred a few radians or a few seconds prior to the standard ride. It is important that both of these ideas come out in the discussion of questions 3 and 4, and that the two transformation expressions $a\sin(bt + c) + d$ and $a\sin(b(t + c)) + d$ are related to each other algebraically. Use student work, if possible, to bring out these two forms, or propose changing one form into the other algebraically and then having students discuss how each form tells the story of the horizontal transformation differently.

End the discussion by connecting equations of the form $y = a\sin(b(x \pm c)) + d$ to a graph, by noting the horizontal transformation represented by the parameter $\pm c$, and relate this shift of the graph to the context to help solidify why the graph moves left or right, based on whether we add or subtract a few seconds represented by the parameter *c.* Help students notice that when the equation is written in this form the horizontal transformation is easier to identify, and the effect of the parameter added inside the function is consistent with horizontal transformations we have noted in other functions.

Aligned Ready, Set, Go: Trigonometric Functions, Equations and Identities 7.3

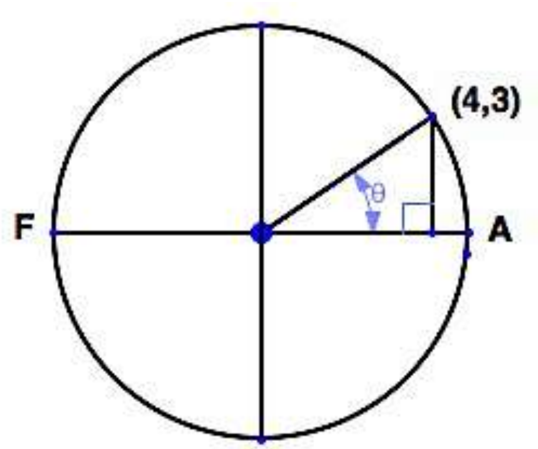
READY, SET, GO! Name Period Date

READY

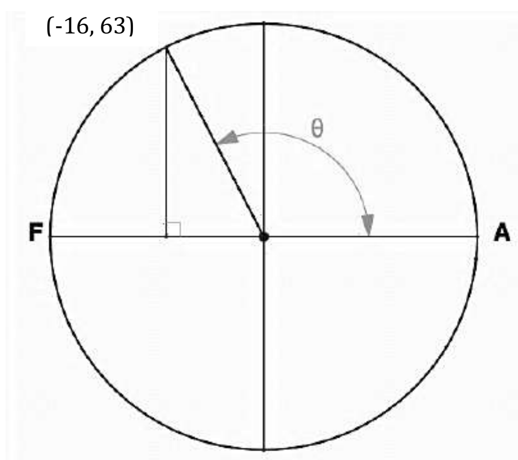
Topic: Using the definition of tangent

Use what you know about the definition of tangent in a right triangle to find the value of tangent θ for each of the right triangles below.

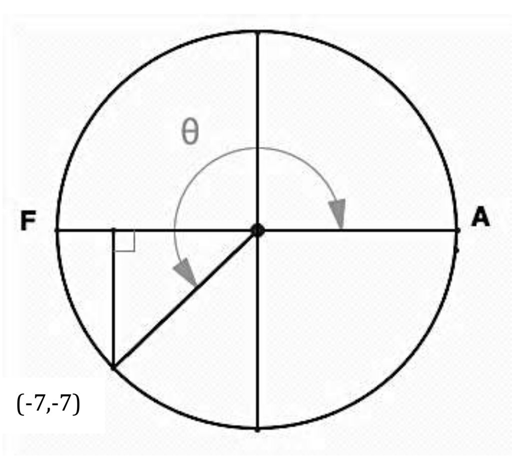
1. $\tan\theta =$



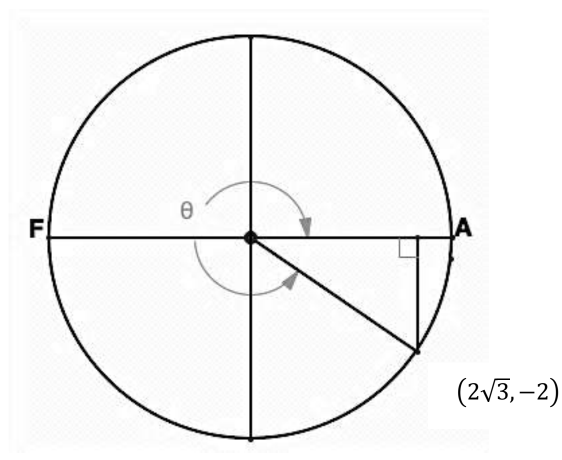
2. $\tan\theta =$



3. $\tan\theta =$



4. $\tan\theta =$



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5. In each graph, the angle of rotation is indicated by an arc and θ. Describe the angles of rotation from $0\ to\ 2\pi$ that make tangent θ be positive and the angles of rotation that make tangent θ be negative.

SET

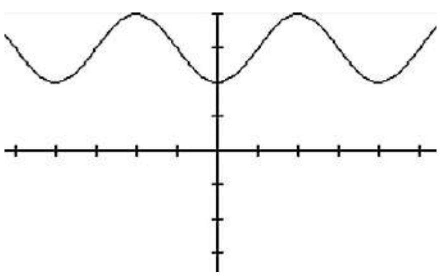
Topic: Connecting trig graphs with their equations

Match each trigonometric representation on the left with an equivalent representation on the right. Then check your answers with a graphing calculator. (The scale on the vertical axis is 1. The scale on the horizontal axis is $\left(\frac{\pi}{2}\right)$.)

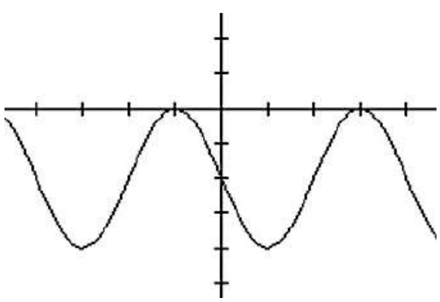
6. $y = -3sin\left(\theta + \frac{\pi}{2}\right)$

7. $y = 3\cos\left(\theta + \frac{\pi}{2}\right)$

8.



9.



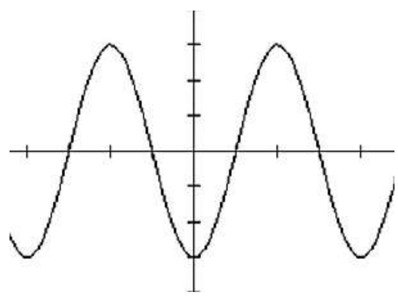
10. $y = \sin\left(2\left(\theta + \frac{\pi}{2}\right)\right) - 2$

11. $y = \sin(x + \pi)$

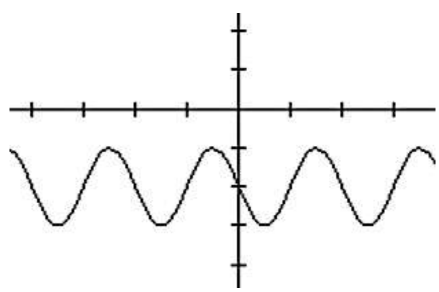
A. $y = -3\sin\theta$

B. $y = -sin\theta$

C.



D.



E. $y = 2cos\left(\theta + \frac{\pi}{2}\right) - 2$

F. $y = \cos(x + \pi) + 3$

12. Choose the equation(s) that has the same graph as $y = cos\theta$.

a. $y = \cos(-\theta)$

b. $y = \cos(\theta - \pi)$

Use the unit circle to explain why they are the same.

13. Choose the equation(s) that has the same graph as $y = -sin\theta$.

a. $y = \sin(\theta + \pi)$

b. $y = \sin(\theta - \pi)$

Use the unit circle to explain why they are the same.

For each function, identify the amplitude, period, horizontal shift, and vertical shift.

14. $f(t) = 150\cos\left(\frac{\pi}{6}(t - 8)\right) + 80$

amplitude:

period:

horizontal shift:

vertical shift:

15. $f(t) = 4.5\sin\left(\frac{\pi}{4}t + \frac{3}{4}\right) + 8$

amplitude:

period:

horizontal shift:

vertical shift:

GO

Topic: Making sense of composite trig functions

Recall that a composite function places one function such as g(x), inside the other, f(x), by replacing the x in f(x) with the entire function g(x). In general, the notation is $f\left(g(x)\right)$. Also, recall that inverse functions "undo" each other. Since, $sin^{-1}\left(\frac{1}{2}\right)$ is an angle of $30°$ because $sin\,30° = \frac{1}{2}$ the composite $sin\left(sin^{-1}\frac{1}{2}\right)$ is simply $\frac{1}{2}$. Sine function "undoes" what $\sin^{-1}\theta$ was does.

Not all composite functions are inverses. Note: problems18 – 24.

Answer the following.

16. $\sin\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$

17. $\cos\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$

18. $\tan(\tan^{-1} 9.52)$

19. $\sin\left(\cos^{-1}\frac{1}{2}\right)$

20. $\cos(\tan^{-1} 1)$

21. $\sin(\tan^{-1} 2.75)$

22. $\cos(\sin^{-1} 1)$

23. $\cos(\tan^{-1} 0)$

24. $\sin(\tan^{-1} undefined)$

7.4H Off on a Tangent

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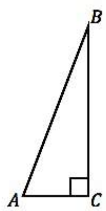
A Develop and Solidify Understanding Task

Recall that the right triangle definition of the tangent ratio is:



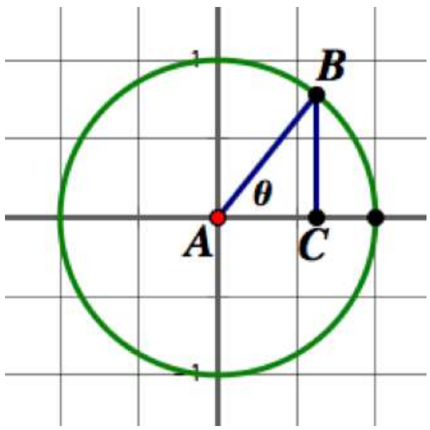
$$\tan(A) = \frac{length\ of\ side\ opposite\ angle\ A}{length\ of\ side\ adjacent\ to\ angle\ A}$$

1. Revise this definition to find the tangent of any angle of rotation, given in either radians or degrees. Explain why your definition is reasonable.

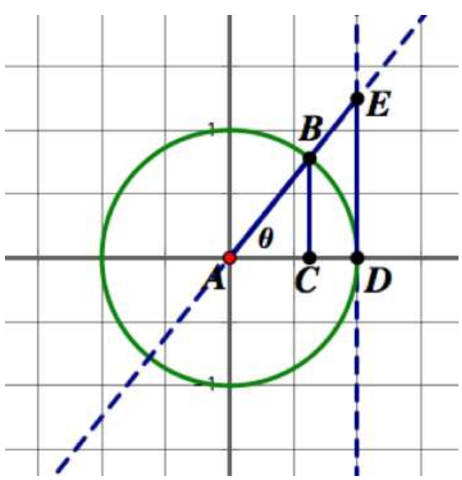
2. Revise this definition to find the tangent of any angle of rotation drawn in standard position on the unit circle. Explain why your definition is reasonable.

We have observed that on the unit circle the value of sine and cosine can be represented with the length of a line segment.

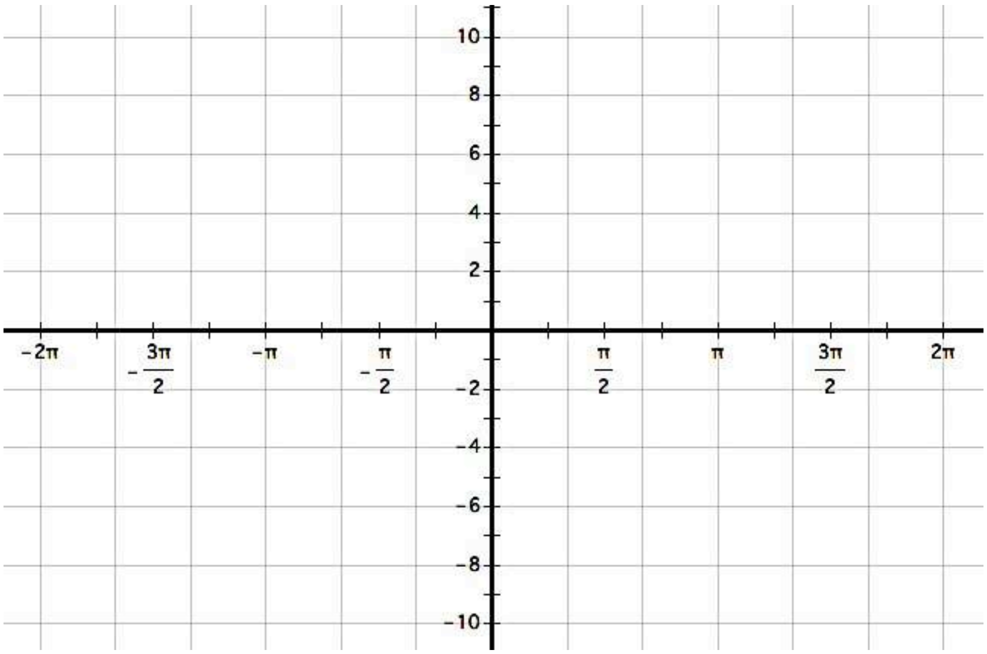
3. Indicate on the following diagram which segment's length represents the value of $\sin(\theta)$ and which represents the value of $\cos(\theta)$ for the given angle θ.



There is also a line segment that can be defined on the unit circle so that its length represents the value of $\tan(\theta)$. Consider the length of $\overline{DE}$ in the unit circle diagram below. Note that ΔADE and ΔABC are right triangles. Write a convincing argument explaining why the length of segment DE is equivalent to the value of $\tan(\theta)$ for the given angle θ.



4. On the coordinate axes below sketch the graph of $y = \tan(\theta)$ by considering the length of segment DE as θ rotates through angles from 0 radians to 2π radians. Explain any interesting features you notice in your graph.



Extend your graph of $y = \tan(\theta)$ by considering the length of segment DE as θ rotates through negative angles from 0 radians to -2π radians.

5. Using your unit circle diagrams from the task *Water Wheels and the Unit Circle*, give exact values for the following trigonometric expressions:

a. $\tan\left(\frac{\pi}{6}\right) =$

b. $\tan\left(\frac{5\pi}{6}\right) =$

c. $\tan\left(\frac{7\pi}{6}\right) =$

d. $\tan\left(\frac{\pi}{4}\right) =$

e. $\tan\left(\frac{3\pi}{4}\right) =$

f. $\tan\left(\frac{11\pi}{6}\right) =$

g. $\tan\left(\frac{\pi}{2}\right) =$

h. $\tan(\pi) =$

i. $\tan\left(\frac{7\pi}{3}\right) =$

Functions are often classified based on the following definitions:

- A function $f(x)$ is classified as an **odd function** if $f(-\theta) = -f(\theta)$
- A function $f(x)$ is classified as an **even function** if $f(-\theta) = f(\theta)$

6. Based on these definitions and your work in this module, determine how to classify each of the following trigonometric functions.

   - The function $y = sin(x)$ would be classified as an [odd function, even function, neither an odd or even function]. Give evidence for your response.

   - The function $y = cos(x)$ would be classified as an [odd function, even function, neither an odd or even function]. Give evidence for your response.

   - The function $y = \ tan(x)$ would be classified as an [odd function, even function, neither an odd or even function]. Give evidence for your response.

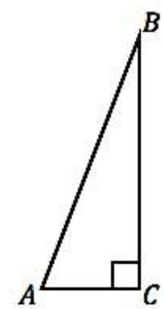
Extra for Experts:

When defining the trig ratios using right triangles we named possible ratios of sides, such as the sine ratio defined as the ratio of *the length of the side opposite the acute angle* to *the length of the hypotenuse*, the cosine ratio defined as the ratio of *the length of the side adjacent to the acute angle* to *the length of the hypotenuse*, and the tangent ratio defined as *the length of the side opposite the acute angle* to *the length of the side adjacent to the acute angle.*

It is sometimes useful to consider the reciprocals of these ratios, leading to the definition of three additional trignometric ratios: *secant, cosecant* and *cotangent,* as defined below.

The secant ratio:

$$\sec(A) = \frac{length\ of\ the\ hypotenuse}{length\ of\ side\ adjacent\ to\ angle\ A}$$



The cosecant ratio:

$$\csc(A) = \frac{length\ of\ the\ hypotenuse}{length\ of\ side\ opposite\ angle\ A}$$

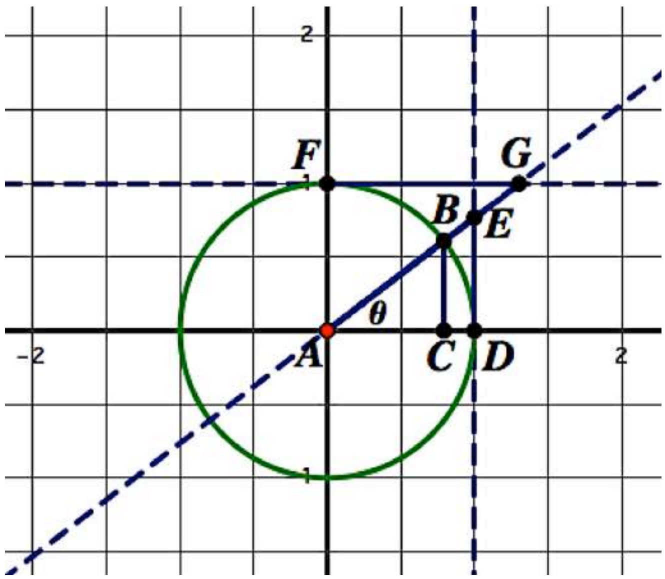
The cotangent ratio:

$$\cot(A) = \frac{length\ of\ side\ adjacent\ to\ angle\ A}{length\ of\ side\ opposite\ angle\ A}$$

7. Complete the following statements:

a. The ____________ ratio is the reciprical of the sine ratio.
b. The ____________ ratio is the reciprical of the cosine ratio.
c. The ____________ ratio is the reciprical of the tangent ratio.
d. $\frac{1}{\cos\theta} =$ $\qquad$ $\frac{1}{\sin\theta} =$ $\qquad$ $\frac{1}{\tan\theta} =$
e. $\frac{1}{\cot\theta} =$ $\qquad$ $\frac{1}{\sec\theta} =$ $\qquad$ $\frac{1}{\csc\theta} =$

There are also line segments that can be defined on the unit circle so that their lengths represents the value of $\sec(\theta)$, $\csc(\theta)$, or $\cot(\theta)$. Consider the lengths of $\overline{AE}, \overline{AG}$ and $\overline{FG}$ in the unit circle diagram below. Note that ΔADE and ΔAFG are right triangles and $\angle FGA \cong \angle DAG$ since they are alternate interior angles formed by parallel lines $\overleftrightarrow{FG}$ and $\overleftrightarrow{AD}$ intersected by transversal $\overleftrightarrow{AG}$.

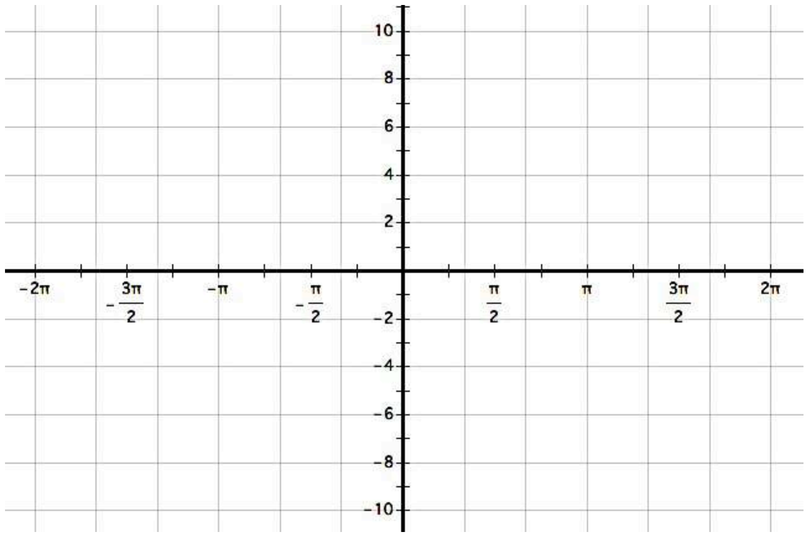


8. Which segment has a length that would be equal to $\sec(\theta)$? Explain how you know.

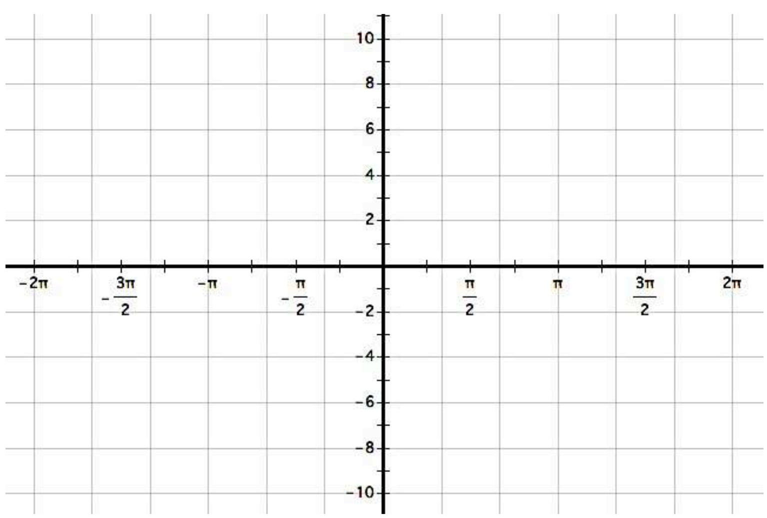
9. Which segment has a length that would be equal to $\csc(\theta)$? Explain how you know.

10. Which segment has a length that would be equal to $\cot(\theta)$? Explain how you know.

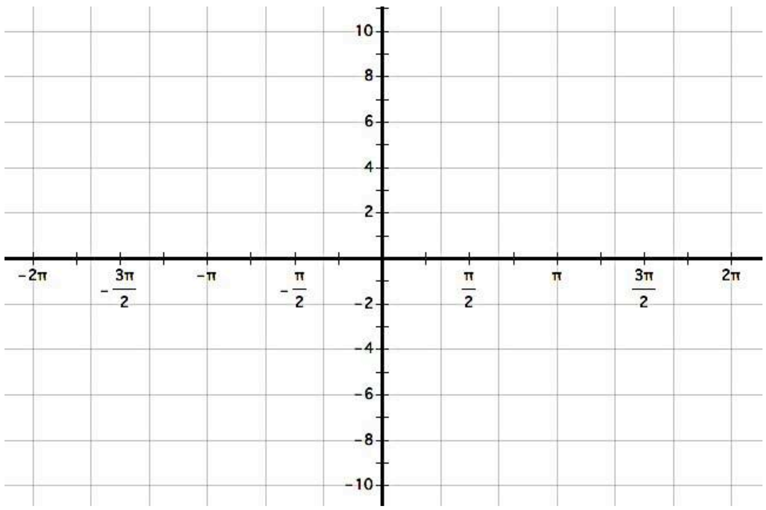
11. On the coordinate axes below sketch the graph of $y = \sec(\theta)$ by considering the length of its corresponding segment in the unit circle diagram above as θ rotates through angles from 0 radians to 2π radians, and from 0 radians to -2π radians. Explain any interesting features you notice in your graph.



12. On the coordinate axes below sketch the graph of $y = \csc(\theta)$ by considering the length of its corresponding segment in the unit circle diagram above as θ rotates through angles from 0 radians to 2π radians, and from 0 radians to -2π radians. Explain any interesting features you notice in your graph.



13. On the coordinate axes below sketch the graph of $y = \cot(\theta)$ by considering the length of its corresponding segment in the unit circle diagram above as θ rotates through angles from 0 radians to 2π radians, and from 0 radians to -2π radians. Explain any interesting features you notice in your graph.



7.4 Off on a Tangent – Teacher Notes

A Develop and Solidify Understanding Task

Purpose: The purpose of this task is to extend the definition of the tangent from the right triangle trigonometric ratio definition—$\tan(A) = \frac{length\ of\ side\ opposite\ angle\ A}{length\ of\ side\ adjacent\ to\ angle\ A}$—to an angle of rotation definition: $\tan(\theta) = \frac{y}{x}$. The graph of the tangent function is obtained by representing the tangent of an angle of rotation by the length of a line segment related to the unit circle, and tracking the length of the line segment as the angle of rotation increases around the unit circle. The trigonometric identity $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ is also explored in terms of the unit circle.

Core Standards Focus:

F.TF.2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

F.TF.3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for **π**/3, **π**/4 and **π**/6, and use the unit circle to express the values of sine, cosine, and tangent for π–x, π+x, and 2π–x in terms of their values for x, where x is any real number.

F.TF.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

Related Standards: F.IF.4, F.IF.7, F.IF.9

Standards for Mathematical Practice:
SMP 7 – Use appropriate tools strategically

Vocabulary: Students will define the *tangent function* for an angle of rotation as $\tan\theta = \frac{y}{x}$ where x and y are the coordinates of a point on a circle where the terminal ray of the angle of rotation intersects the circle when the angle is drawn in standard position (i.e., the vertex of the angle is at the origin and the initial ray lies along the positive x-axis.) The will define the *secant function* as the reciprocal of the cosine function, $y = \frac{1}{\cos\theta}$, and as $\sec\theta = \frac{r}{x}$; the *cosecant function* as the reciprocal of the sine function, $y = \frac{1}{\sin\theta}$, and as $\csc\theta = \frac{r}{y}$; and the *cotangent function* as the reciprocal of the tangent function, $y = \frac{1}{\tan\theta}$, and as $\cot\theta = \frac{x}{y}$.

The Teaching Cycle:
Launch (Whole Class):
Remind students that we have redefined sine and cosine for angles of rotation drawn in standard position by using the values of x, y and r. Ask how they might redefine tangent using these same values. Students should note that the definition $\tan(\theta) = \frac{y}{x}$ is independent of the value of r.

Examine the two unit circle drawings in question 3 together as a class. In the drawings label segment AC as $x = \cos(\theta)$, segment BC as $y = \sin(\theta)$ and segment AB as $r = 1$. In the second drawing note that ΔABC is similar to ΔADE and that the measure of segment AE is 1. Using this information ask students to consider what this implies about the measure of segment DE. Give students a couple of minutes to suggest that since the triangles are similar they can write the proportion $\frac{DE}{AE} = \frac{BC}{AC}$ or $\frac{DE}{1} = \frac{y}{x}$. They should recognize that the length of segment DE is defined in the same way that we have defined $\tan(\theta)$. That is, the length of segment DE represents the value of $\tan(\theta)$ in the same way that the length of segment AC represents the value of $\cos(\theta)$ and the length of segment BC represents the value of $\sin(\theta)$. You may also want to point out that the trigonometric identity $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ is present in this diagram.

Now that we have a way of visually representing the magnitude of the value of $\tan(\theta)$, assign students to work on determining what this implies about the shape and features of the graph of $y = \tan(\theta)$. Also have them work on the rest of the task by using their unit circle diagrams.

Explore (Small Group):

If students are having a hard time sketching the graph, focus their attention on small intervals of θ. For example, what happens to the length of segment DE as θ increases from 0 radians to $\frac{\pi}{2}$ radians? What happens when $\theta = \frac{\pi}{2}$? What happens when θ increases from $\frac{\pi}{2}$ to π? How would you draw ΔADE on this interval? What about negative angles of rotation?

Watch as students compute values of $tan(\theta)$ using information recorded on their unit circle diagrams. Students may need help simplifying the ratios formed by $\frac{y}{x}$. Allow students to leave these ratios unsimplified until the whole class discussion when you can discuss some of the arithmetic involved, hopefully by using work from students who are successful at simplifying these ratios. Look for such students.

Listen for how students apply the definitions of odd and even functions to the sine, cosine and tangent functions. What representations do they draw upon to make these decisions: the symmetry of points around the unit circle, a graph of the function, or some other ways of reasoning?

Discuss (Whole Class):

Focus the whole class discussion on the following items:

- The graph of the tangent function, including the period of π and the behavior of the graph near and at $\pm\frac{\pi}{2}$ and $\pm\frac{3\pi}{2}$ (the vertical asymptotes).
- The values of the tangent function at angles that are multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$, including the arithmetic of simplifying these ratios.

- The classification of sine, cosine and tangent as even or odd functions and the evidence used to support these classifications (e.g., the graph of the function or the symmetry of the unit circle).
- The definitions and graphs of the secant function, the cosecant function, and the cotangent function.
- How does the idea of reciprocals show up in the graphs of the reciprocal functions? That is, if we graph, for example, $y = \sin\theta$ and $y = \csc\theta$ on the same graph, how do the reciprocal values show up in the graphs? For example, the reciprocal of 0 is undefined, so whenever the graph of the sine is 0, the graph of the cosecant will have a vertical asymptote. Ask students to describe how thinking about reciprocals could be a useful curve-sketching tool when graphing the secant, cosecant and cotangent graphs.

Aligned Ready, Set, Go: Trigonometric Functions, Equations and Identities 7.4

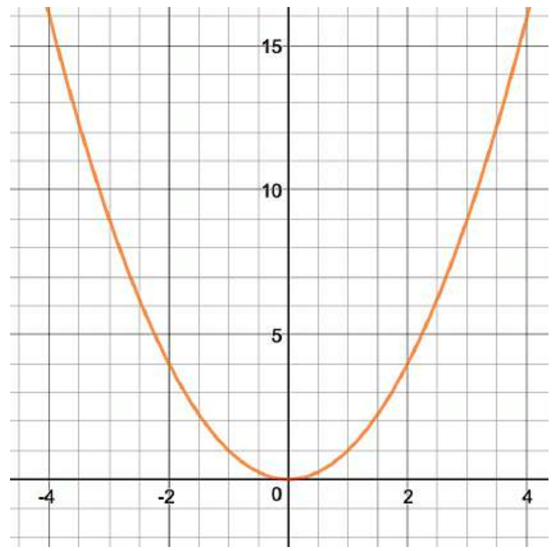
READY, SET, GO! Name Period Date

READY

Topic: Making rigid and non-rigid transformations on functions

The equation of a parent function is given. Write a new equation with the given transformations. Then sketch the new function on the same graph as the parent function. (If the function has asymptotes, sketch them in.)

1. $y = x^2$



Vertical shift: up 8

horizontal shift: left 3

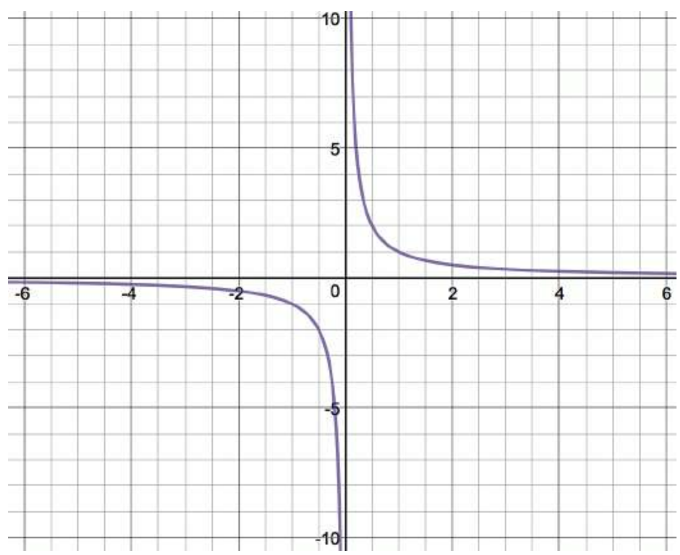
dilation: ¼

Equation:

Domain:

Range:

2. $y = \frac{1}{x}$



Vertical shift: up 4

horizontal shift: right 3

dilation: -1

Equation:

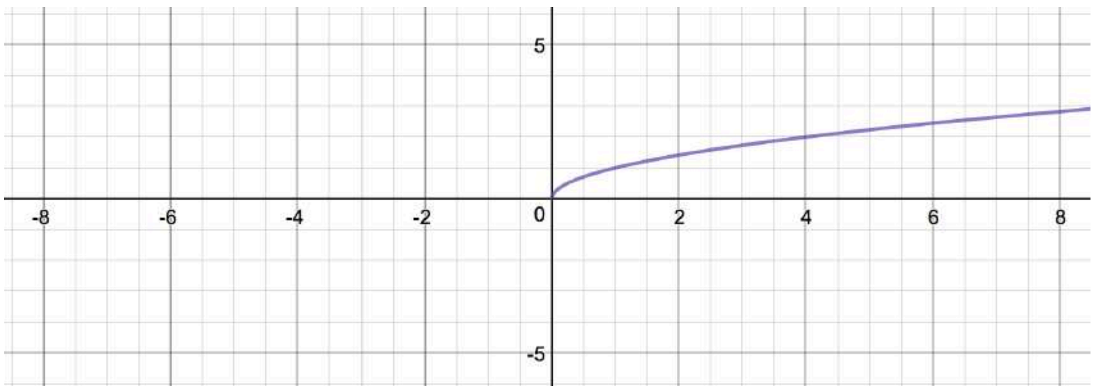
Domain:

Range:

3. $y = \sqrt{x}$

Vertical shift: none. horizontal shift: left 5 dilation: 3

Equation: Domain: Range:



4. $y = \sin x$

Vertical shift: 1

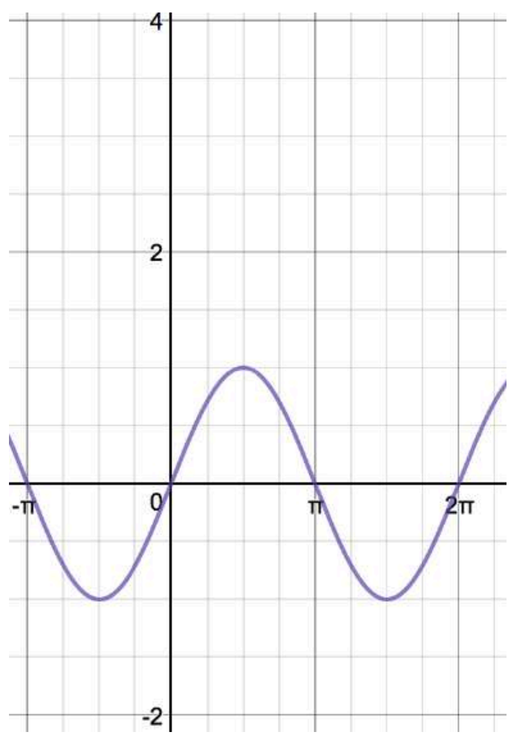
horizontal shift: left $\frac{\pi}{2}$

dilation (amplitude): 3

Equation:

Domain:

Range:

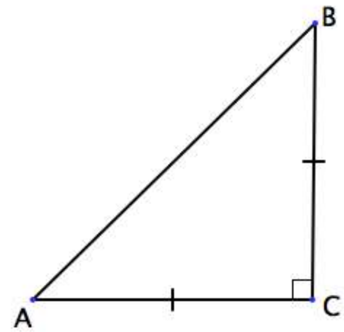


SET

Topic: Connecting values in the special triangles with radian measures

5. Triangle ABC is a right triangle. AB = 1.

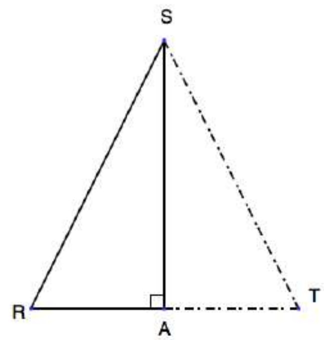
   Use the information in the figure to label the length of the sides and measure of the angles.



6. Triangle RST is an equilateral triangle. RS = 1
   $\overline{SA}$ is an altitude

   Use the information in the figure to label the length of the sides, the length of $\overline{RA}$, and the *exact* length of $\overline{SA}$.

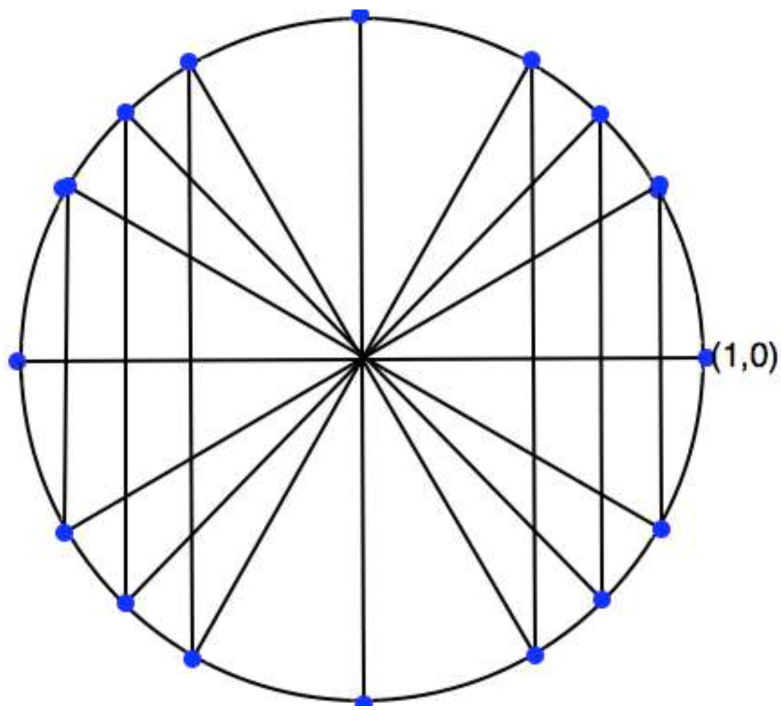
   Label the measure of angles RSA and SRA.



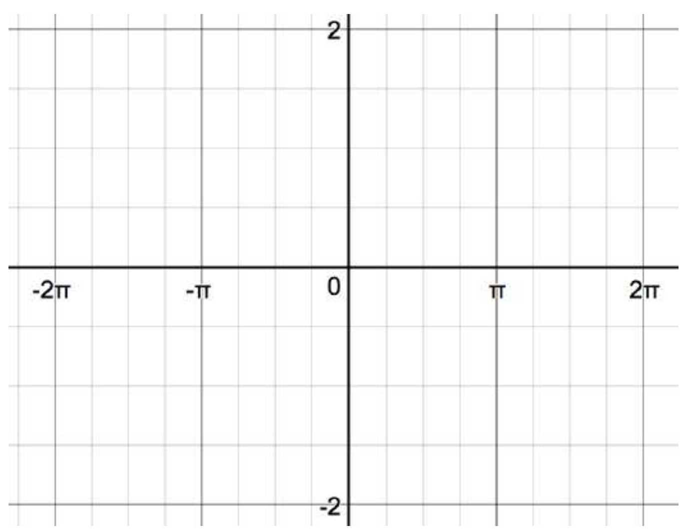
7. Use what you know about the unit circle and the information from the figures in problems 6 and 7 to fill in the table. Some values will be undefined.

function	$\theta = \frac{\pi}{6}$	$\theta = \frac{\pi}{4}$	$\theta = \frac{\pi}{3}$	$\theta = \frac{\pi}{2}$	$\theta = \pi$	$\theta = \frac{3\pi}{2}$	$\theta = 2\pi$
$\sin\theta$							
$\cos\theta$							
$\tan\theta$							
$\csc\theta$							
$\sec\theta$							
$\cot\theta$							

8. Label all of the points and angles of rotation in the given unit circle.



9. **Graph $f(x) = \tan\theta$.** Use your table of values above for $f(x) = \tan\theta$. Sketch your asymptotes with dotted lines.



10. Where do asymptotes always occur?

11. How can you use the sine function to determine the location of the asymptotes for cosecant?

GO

Topic: Recalling trig facts

Answer the questions below. Be sure you can justify your thinking.

12. Given triangle ABC with angle C being a right angle, what is the sum of $m\angle A + m\angle B$?

13. Identify the quadrants in which $\sin\theta\ is\ positive$.

14. Identify the quadrants in which $\cos\theta\ is\ negative$.

15. Identify the quadrants in which $\tan\theta\ is\ positive$.

16. Explain why it is impossible for $\sin\theta > 1$.

17. Name the angles of rotation (in radians) for when $\sin\theta = \cos\theta$.

18. For which trig functions do a positive rotation and a negative rotation always give the same value?

19. Explain why in the unit circle $\tan\theta = \frac{y}{x}$.

20. Which function gives the slope of the hypotenuse in a right triangle?

21. Explain why $\sin\theta = \cos(90° - \theta)$.

22. Write the Pythagorean Identity and then prove it.

23. Name the trigonometric function(s) that are even functions.

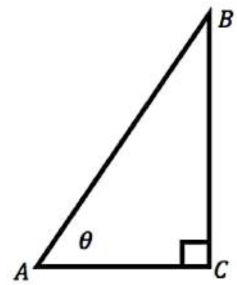
7.5 Maintaining Your Identity

A Develop Understanding Task

Right triangles and the unit circle provide images that can be used to derive, explain and justify a variety of trigonometric identities.

1. For example, how might the right triangle diagram at the right help you justify why the following identity is true for all angles θ between 0° and 90°?

   $\sin(\theta) = \cos(90° - \theta)$

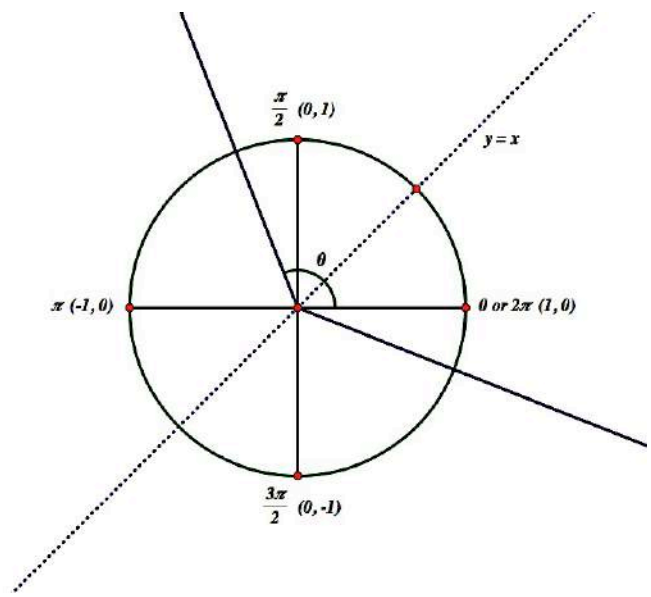


2. Since we have extended our definition of the sine to include angles of rotation, rather than just the acute angles in a right triangle, we might wonder if this identity is true for all angles θ, not just those that measure between 0° and 90°?

   A version of this identity that uses radian rather than degree measure would look like this:

   $$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$$

   How might you use this unit circle diagram to justify why this identity is true for all angles θ?



Fundamental Trig Identities

3. Here are some additional trig identities. Use either a right triangle diagram or a unit circle diagram to justify why each is true.

   a. $\sin(-\theta) = -\sin(\theta)$

   b. $\cos(-\theta) = \cos(\theta)$

   c. $\sin^2\theta + \cos^2\theta = 1$ [Note: This is the preferred notation for $(\sin\theta)^2 + (\cos\theta)^2 = 1$]

   d. $\frac{\sin\theta}{\cos\theta} = \tan\theta$

4. Use right triangles or a unit circle to help you form a conjecture for how to complete the following statements as trig identities. How might you use graphs to gain additional supporting evidence that your conjectures are true?

   a. $\sin(\pi - \theta) =$ ____________________ and $\cos(\pi - \theta) =$ ____________________

   b. $\sin(\pi + \theta) =$ ____________________ and $\cos(\pi + \theta) =$ ____________________

   c. $\sin(2\pi - \theta) =$ __________________ and $\cos(2\pi - \theta) =$ __________________

5. We can use algebra, along with some fundamental trig identities, to prove other identities. For example, how can you use algebra and the identities listed above to prove the following identities?

   a. $\tan(-\theta) = -\tan(\theta)$

   b. $\tan(\pi + \theta) = \tan(\theta)$

7.5 Maintaining Your Identity – Teacher Notes

A Develop Understanding Task

Purpose: The unit circle and right triangle trigonometry allow students to derive a variety of useful trigonometric identities. The purpose of this task is to practice using diagrams to extend the students' repertoire of identities to include identities related to the odd and even behavior of the sine and cosine functions and the Pythagorean identity.

Core Standards Focus:

F.TF.3 (+) Use the unit circle to express the values of sine, cosine, and tangent for $\pi - x$, $\pi + x$, and $2\pi - x$ in terms of their values for x, where x is any real number.

F.TF.4 (+) Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

F.TF.8 Prove the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ and use it to find $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ given $\sin(\theta)$, $\cos(\theta)$, or $\tan(\theta)$ and the quadrant of the angle.

Standards for Mathematical Practice:

SMP 7 – Look for and make use of structure

Vocabulary: Students will be introduced to the concept of *trigonometric identities*, statements that are true for all values of θ.

The Teaching Cycle:

Launch (Whole Class):

Use the identity on the first page of the task in both its right triangle/degree form and its unit circle/radian form to launch the discussion on using a diagram to justify trig identities. Allow students a few minutes to individually examine the two diagrams and relate them to the notation in the two versions of the identity. You might ask questions to prompt thinking such as, "Given the marked angle θ, where would the angle $90° - \theta$, (or the angle $\frac{\pi}{2} - \theta$), appear in the diagram?" Let a couple of students share their reasoning, based on the diagram, as to why the sine of one of these angles is equal to the cosine of the other. For the right triangle diagram students should notice that the ratio of sides defining the sine of one of the angles is the same ratio of sides that defines the cosine of the other angle. For the unit circle diagram they should note that the terminal ray of the angle that starts at 0 and rotates counterclockwise θ radians and the terminal ray of the angle that starts at $\pi/2$ and rotates clockwise θ radians are mirror images about the line $y = x$. Therefore, the x and y coordinates of the points where these terminal rays intersect the unit circle are switched. Since these coordinates define the magnitudes of the sine and cosine of the angles on the unit circle, the sine of one of the angles is the same as the cosine of the other.

Inform students that they are to continue to use right triangles or unit circle diagrams to explore, illustrate and justify each of the statements listed in the remainder of the task.

Explore (Small Group):
Questions 3a and 3b can best be illustrated and justified on a unit circle diagram, since the identity requires students to think about the relationship between angles drawn in both positive and negative directions, which would not make sense in a right triangle.

Question 3c can be justified on a right triangle using the Pythagorean theorem, or on the unit circle using the equation of a circle of radius 1.

Question 3d can be justified using ratios of sides on a right triangle, or using the angle of rotation definitions of sine, cosine and tangent based on the x and y coordinates of points on the unit circle.

Note that questions 4a-4d ask students to conjecture an identity by completing the equality statements, and that a graph can be used as additional support for exploring or justifying students' conjectures. If students are having difficulty conjecturing how to complete one of the identities, suggest that they graph the given expression as a function. That is, graphing $y = \sin(\pi - x)$ can lead students to conjecture that $\sin(\pi - x) = \sin(x)$.

Questions 5a and 5b ask students to make an algebraic argument, rather than a geometric one. These problems can be used to extend the work of the task for some students, while allowing time for other students to finish questions 1-4. When a few students have finished questions 5a and 5b, or if students are struggling with the algebra, move to the whole class discussion.

Discuss (Whole Class):
Select students to present their justifications of the identities in questions 1-3 and their conjectures and justifications for the identities in questions 4. If there are justifications that are based on both a right triangle diagram and a unit circle diagram you might want to have both justifications presented in order to reinforce that these identities are true—where applicable—using both the "right triangle ratio of sides" trig definition and the "coordinates of points on a unit circle" trig definition of sine and cosine.

Conclude the discussion by having students present their algebraic justifications of the two statements in question 5, or work through the algebra of statement 5a together as a class and then give students time to work through statement 5b on their own. The essence of proving these statements about tangent functions is to use the identity given in statement 3d to turn the tangent into the ratio of the sine and cosine, and then apply identities we have written for the sine and cosine. This algebra work will give students a reason for creating a collection of identities in the first place, and will prepare students for some of the algebraic work that will occur in the following task.

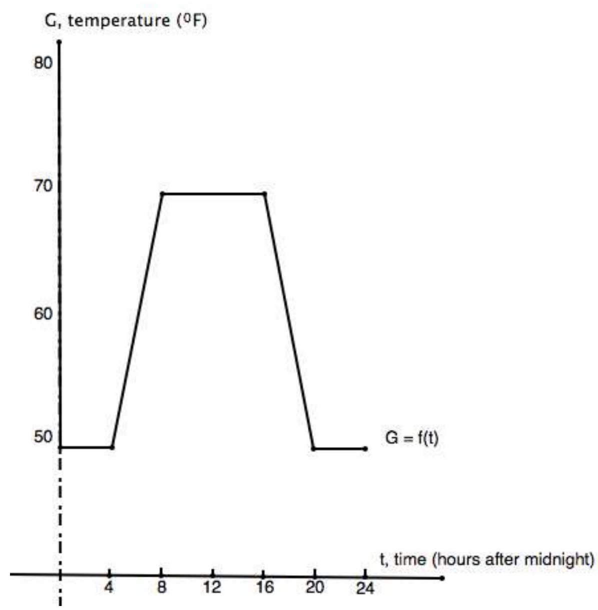
Aligned Ready, Set, Go: Trigonometric Functions, Equations and Identities 7.5

READY, SET, GO! Name Period Date

READY

Topic: Modeling transformations

A school building is kept warm only during school hours, in order to save money. Figure 6.1 shows a graph of the temperature, G, in °F, as a function of time, t, in hours after midnight. At midnight (t = 0), the building's temperature is 50 °F. This temperature remains the same until 4 AM. Then the heater begins to warm the building so that by 8 am the temperature is 70°F. That temperature is maintained until 4 pm, when the building begins to cool. By 8 pm, the temperature has returned to 50°F and will remain at that temperature until 4 am.



1. In January many students are sick with the flu. The custodian decides to keep the building 5 °F warmer. Sketch the graph of the new schedule on figure 6.1.

2. If $G = f(t)$ is the function that describes the original temperature setting, what would be the function for the January setting?

3. In the spring, the drill team begins early morning practice. The custodian then changes the original setting to start 2 hours earlier The building now begins to warm at 2 am instead of 4 am and reaches 70 °F at 6 am. It begins cooling off at 2 pm instead of 4 pm and returns to 50 °F at 6 pm instead of 8 pm. Sketch the graph of the new schedule on figure 6.1.

4. If $G = f(t)$ is the function that describes the original temperature setting, what would be the function for the spring setting?

SET

Topic: Using trigonometric identities to find additional trig values

The *Cofunction identity* states: $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ *and* $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$

Complete the statements, using the *Cofunction identity.*

5. $\text{Sin } 70° = \cos$ ______°
6. $\text{Sin } 28° = \cos$ ______°
7. $\text{Cos } 54° = \sin$ ______°
8. $\text{Sin } 9° = \cos$ ______°
9. $\text{Cos } 72° = \sin$ ______°
10. $\text{Cos } 45° = \sin$ ______°
11. $\text{Cos } \frac{\pi}{8} = \sin$ ___
12. $\text{Sin } \frac{5\pi}{12} = \cos$ ___
13. $\text{Sin } \frac{3\pi}{10} = \cos$ ___

14. Let $\sin\theta = \frac{3}{4}$.

a) Use the *Pythagorean identity* $(sin^2\theta + cos^2\theta = 1)$, to find the value of $\cos\theta$.

b) Use the *Quotient identity* $\left(\tan\theta = \frac{\sin\theta}{\cos\theta}\right)$, the given information, and your answer in part (a) to calculate the value of $\tan\theta$.

15. Let $\cos\beta = \frac{12}{13}$.

a) Find $\sin\beta$. Use the *Pythagorean identity* $(sin^2\theta + cos^2\theta = 1)$.

b) Find $\tan\beta$. Use the *Quotient identity* $\left(\tan\theta = \frac{\sin\theta}{\cos\theta}\right)$.

c) Find $\cos\left(\frac{\pi}{2} - \theta\right)$. Use a *Cofunction identity.*

Use trigonometric definitions and identities to prove the statements below.

16. $\tan\theta\cos\theta = \sin\theta$

17. $(1+\cos\beta)(1-\cos\beta) = \sin^2\beta$

18. $(1+\sin\alpha)(1-\sin\alpha) = \cos^2\alpha$

19. $sin^2W - cos^2W = 2sin^2W - 1$

GO

Topic: Solving simple trig equations using the special angle relationships

Find <u>two</u> solutions of the equation. Give your answers in degrees $(0° \leq \theta \leq 360°)$ and radians $(0 \leq \theta \leq 2\pi)$. Do NOT use a calculator.

20. $\sin\theta = \frac{1}{2}$

degrees: ______________________________

radians: ______________________________

21. $\sin\theta = -\frac{1}{2}$

degrees: ______________________________

radians: ______________________________

22. $\cos\theta = \frac{\sqrt{2}}{2}$

degrees: ______________________________

radians: ______________________________

23. $\sin\theta = -\frac{\sqrt{3}}{2}$

degrees: ______________________________

radians: ______________________________

24. $\tan\theta = -1$

degrees: ______________________________

radians: ______________________________

25. $\tan\theta = \sqrt{3}$

degrees: ______________________________

radians: ______________________________

7.6 Hidden Identities

A Practice Understanding Task

Note: Because trig functions are periodic, trig equations often have multiple solutions. Typically, we are only interested in the solutions that lie within a restricted interval, usually the interval from 0 to 2π. In this task you should find all solutions to the trig equations that occur on $[0, 2\pi]$.

To sharpen their trig skills, Alyce, Javier and Veronica are trying to learn how to solve some trig equations in a math refresher text that they found in an old trunk one of the adults had brought to the archeological site. Here is how each of them thought about one of the problems:

$$\textit{Solve}\text{: } \cos\left(\frac{\pi}{2} - \theta\right) = \frac{1}{2}$$

Alyce: I used the inverse cosine function.

Javier: I first used an identity, and then an inverse trig function. But it was not the same inverse trig function that Alyce used.

Veronica: I graphed $y_1 = \cos\left(\frac{\pi}{2} - \theta\right)$ and $y_2 = \frac{1}{2}$ on my calculator. I seem to have found more solutions.

1. Using their statements as clues, go back and solve the equation the way that each of the friends did.

2. How does Veronica's solutions match with Alyce and Javier's? What might be different?

Solve each of the following trig equations by adapting Alyce and Javier's strategies: that is, you may want to see if the equation can be simplified using one of the trig identities you learned in the previous task; and once you have isolated a trig function on one side of the equation, you can undo that trig function by taking the inverse trig function on both sides of the equation. Once you have a solution, you may want to check to see if you have found all possible solutions on the interval $[0, 2\pi]$ by using a graph as shown in Veronica's strategy.

3. $\sin(-\theta) = -\frac{1}{2}$

4. $\cos\theta \cdot \tan\theta = \frac{\sqrt{3}}{2}$

5. $\sin(2\theta) = \frac{\sqrt{2}}{2}$

7.6 Hidden Identities – Teacher Notes

A Practice Understanding Task

Purpose: In this task, students draw upon some of the identities discovered in the previous task to help them change the form of trigonometric expressions into simpler expressions until the solutions of the equation can be found. The dilemma of finding all of the solutions to an equation when using an inverse trig function to solve a trig equation surfaces again, and students have an opportunity to again consider the meaning of solutions to an equation.

Core Standards Focus:

F.TF.7 (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.*

Standards for Mathematical Practice:

SMP 7 – Look for and make use of structure

Vocabulary: Students will look for all solutions to each trigonometric equation within *restricted intervals* of the variable, such as solving for θ on the interval $[0, 2\pi]$.

The Teaching Cycle:

Launch (Whole Class):

Introduce the equation that Alyce, Javier and Veronica are trying to solve, $\cos\left(\frac{\pi}{2} - \theta\right) = \frac{1}{2}$, and then have students try out each of the three strategies described in the task for solving this equation. Make sure that students are using each of these three uniquely different strategies, and not just solving the problem using a single strategy. After a few minutes, have three students demonstrate each of the three strategies, including Veronica's graphical method, which reveals that this equation has an infinite number of solutions. Have students explain how they could get all of Veronica's solutions from the solutions obtained algebraically using Alyce and Javier's approaches. Then read the statement at the beginning of the task about restricting the solutions to those that occur on the interval $[0, 2\pi]$. Remind students that they should find all solutions within this interval for each trig equation. Students should now be ready to work on problems 3-5 by using a symbolic strategy similar to the work of either Alyce or Javier.

Explore (Small Group):

Encourage students to try to find all solutions to each equation on the interval $[0, 2\pi]$ before they check their algebraic work by using a graph. If the graph reveals more solutions than obtained by the students through their symbolic work, ask them to explain why they might be missing solutions, and how they might look for those solutions without using a graph. This may be particularly necessary on question 5, since two sets of solutions occur on the interval $[0, 2\pi]$ since the period of the trigonometric expression on the left side of the equation has been compressed. Question 5 may create other issues, since the argument of the trig expression is 2θ, instead of just θ.

Discuss (Whole Class):

Discuss each of the three problems thoroughly, listening for misconceptions in students work. For example, in question 3, why can we eliminate the negative signs that appear on both sides of the equation? Don't allow students to just say, "I cancelled out the negative signs on both sides of the equation" or "I got rid of the negative signs by multiplying both sides of the equation by -1." Students need to recognize that the negative sign inside the sine function plays a different role than the negative sign on the fraction on the right side of the equation. They should describe their first step in solving this equation as replacing $\sin(-\theta)$ with $-\sin(\theta)$ based on the identity $\sin(-\theta) = -\sin(\theta)$, and then multiplying both sides of the equation by -1. For question 4, students should use the identity $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ and then recognize that $\frac{\cos(\theta)}{\cos(\theta)} = 1$. On question 5 students will need to find all angles of rotation where the sine value is $\frac{\sqrt{2}}{2}$ during two revolutions counterclockwise around the unit circle, since they will be dividing these values by 2 to solve for θ on the interval $[0, 2\pi]$.

Aligned Ready, Set, Go: Trigonometric Functions, Equations and Identities 7.6

READY, SET, GO! Name Period Date

READY

Topic: Using the calculator to find angles of rotation

Use your calculator and what you know about where sine and cosine are positive and negative in the unit circle to find the two angles that are solutions to the equation. Make sure θ *is always* $0 < \theta \le 2\pi$. Round your answers to 4 decimals.
(Your calculator should be set in radians.)

You will notice that your calculator will sometimes give you a negative angle. That is because the calculator is programmed to restrict the angle of rotation so that the inverse of the function is also a function. Since the requested answers have been restricted to positive rotations, if the calculator gives you a negative angle of rotation, you will need to figure out the positive coterminal angle for the angle that your calculator has given you.

1. $\sin\theta = \frac{4}{5}$
2. $\sin\theta = \frac{2}{7}$
3. $\sin\theta = -\frac{1}{10}$
4. $\sin\theta = -\frac{13}{14}$

5. $\cos\theta = \frac{11}{12}$
6. $\cos\theta = \frac{1}{9}$
7. $\cos\theta = -\frac{7}{8}$
8. $\cos\theta = -\frac{2}{5}$

*Note: When you ask your calculator for the angle, you are "undoing" the trig function. Finding the angle is finding the inverse trig function. When you see "***Find** $\sin^{-1}\left(\frac{4}{5}\right)$" *, you are being asked to find the angle that makes it true. The answer would be the same as the answer your calculator gave you in #1. Another notation that means the inverse sine function is* **arcsin** $\left(\frac{4}{5}\right)$.

SET

Topic: Verifying trig identities with tables, unit circles, and graphs

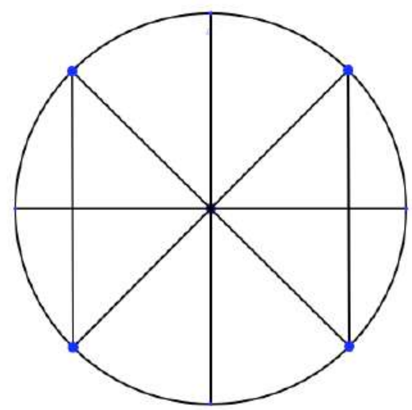
9. **Use the values in the table to verify the Pythagorean identity $\left(\sin^2\theta + \cos^2\theta = 1\right)$. Then use the Quotient Identity $\left(\tan\theta = \frac{\sin\theta}{\cos\theta}\right)$ and the values in the table to write the value of tangent θ.**

9.	$sin\,\theta$	$cos\,\theta$	$sin^2\theta + cos^2\theta = 1$	$tan\theta$
a.	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$		
b.	$\frac{-3}{5}$	$\frac{4}{5}$		
c.	$-\frac{5}{13}$	$-\frac{12}{13}$		
d.	$\frac{\sqrt{14}}{7}$	$\frac{\sqrt{35}}{7}$		
e.	-1	0		
f.	0	1		

10. Label the angles of rotation and the coordinate points around the unit circle on the right. Then use these points to help you fill in the blank.

$cos(\pi - \theta) =$ ______________________

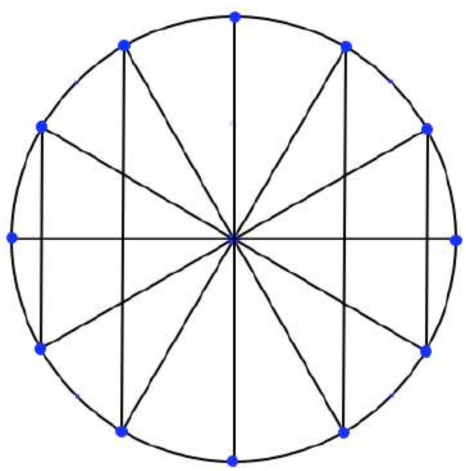
Make your thinking visible by using the diagram. Explain your reasoning.



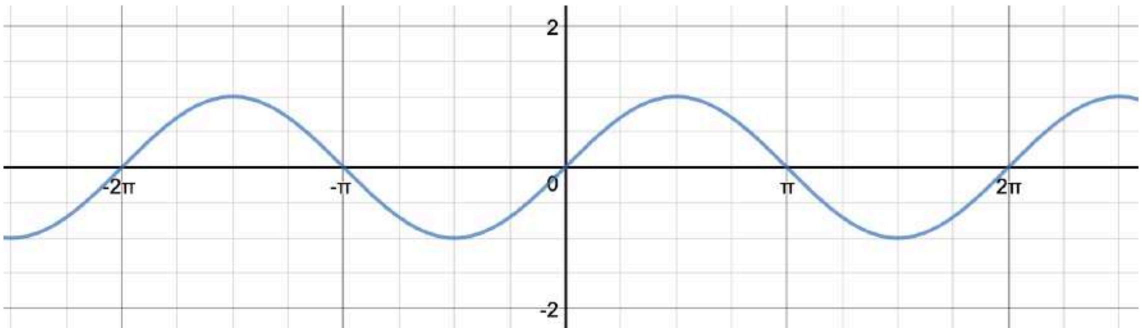
11. Label the angles of rotation and the coordinate points around the unit circle on the right. Then use these points to help you fill in the blank.

$sin(\pi + \theta) =$ ______________________

Make your thinking visible by using the diagram. Explain your reasoning.



12. Use the graph of $\sin\theta$ to help you fill in the blank. $sin(2\pi - \theta) =$ ______________________



Make your thinking visible by using the graph. Explain your reasoning.

GO

Topic: Finding the central angle when given arc length and radius

Find the radian measure of the central angle of a circle of radius *r* that intercepts an arc of length *s*. $(s = r\theta)$

Round answers to 4 decimal places.

Radius	*Arc Length*	*Angle measure in radians*
13. 35 mm	11 mm	
14. 14 feet	9 feet	
15. 16.5 m	28 m	
16. 45 miles	90 miles	

7.7H Double Identity

A Solidify Understanding Task

Sum and Difference Identities

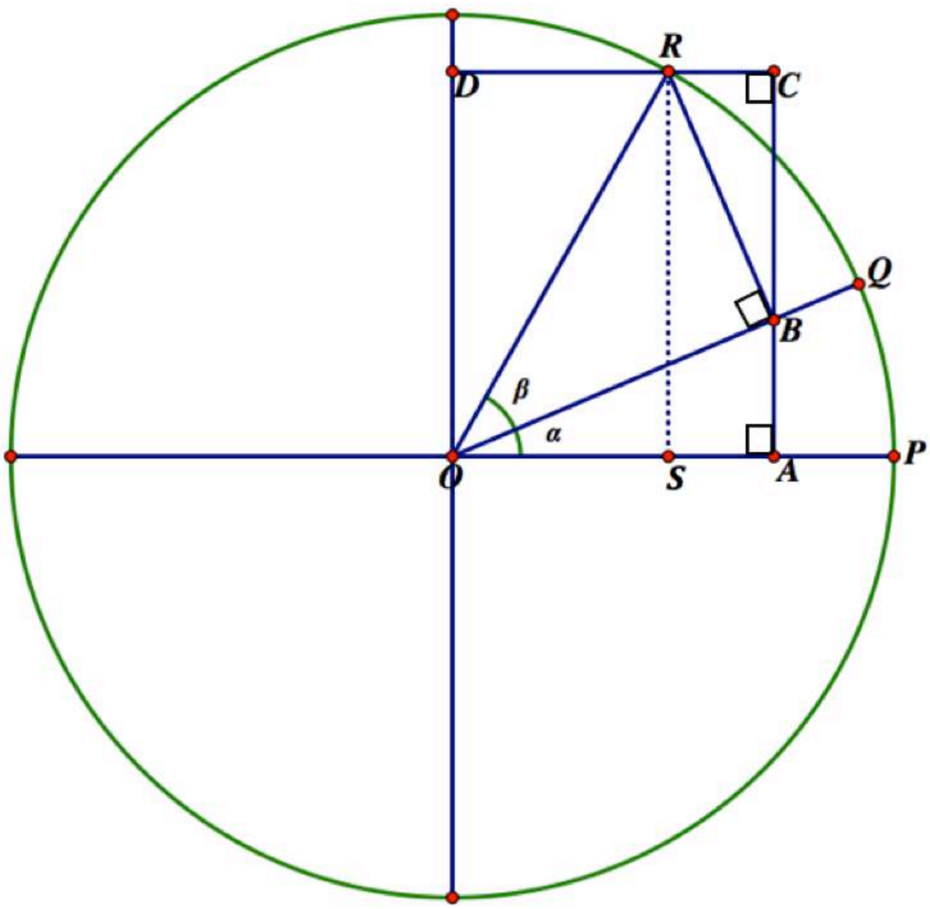
Sometimes it is useful to be able to find the sine and cosine of an angle that is the sum of two consecutive angles of rotation. In the diagram below, point *P* has been rotated α radians counterclockwise around the unit circle to point *Q*, and then point *Q* has been rotated an additional β radians counterclockwise to point *R.* In this task you will examine how the sine and cosine of angle α, angle β and the sum of the two angles, angle α + β, are related?

1. Do you think this is a true statement: $\sin(\alpha+\beta)=\sin\alpha+\sin\beta$

   Why or why not?

Examine the diagram on the following page. Figure *OACD* is a rectangle. Can you use this diagram to state a true relationship that completes this identity? (Your teacher has some hint cards if you need them, but the basic idea is to label all of the segments on the sides of rectangle *OACD* using right triangle trig relationships.) Based on congruent line segments labeled with trigonometric measures in the diagram, complete the following identity:

2. $\sin(\alpha+\beta)=$ ____________________________________



Once you have an identity for $\sin(\alpha + \beta)$ you can find an identity for $\sin(\alpha - \beta)$ algebraically. Begin by noting that $\sin(\alpha - \beta) = \sin(\alpha + (-\beta))$ and apply the identity you found in question #2, along with the identities you explored previously: $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$.

3. $\sin(\alpha - \beta) =$ ______________________________________

You can find an identity for $\cos(\alpha+\beta)$ in the diagram above also. Since $\overline{OA} \cong \overline{DC}$, and $DC = DR + RC$, using trigonometry to determine the lengths of segments OA, DR and RC will reveal this relationship. (Again, your teacher has hint cards if you need them.)

4. $\cos(\alpha+\beta) =$ ________________________________

Now you can also complete this identity using reasoning similar to what you did in question #3.

5. $\cos(\alpha-\beta) =$ ________________________________

The following identities are known as the double angle identities, but they are just special cases of the sum identities you found above.

6. $\sin(2\alpha) = \sin(\alpha+\alpha) =$ ________________________________

7. $\cos(2\alpha) = \cos(\alpha+\alpha) =$ ________________________________

7.7H Double Identity – Teacher Notes

A Solidify Understanding Task

Purpose: The unit circle and right triangle trigonometry allow students to derive a variety of useful trigonometric identities. The purpose of this task is to practice using diagrams to extend the students' repertoire of identities to include identities for the sine and cosine of a sum or difference of angles, and the double angle identities.

Core Standards Focus:

F.TF.9 (+) Prove the addition and subtraction formulas for sine, cosine, and tangent and use them to solve problems.

Standards for Mathematical Practice:

SMP 7 – Look for and make use of structure

Vocabulary: In this task students will be introduced to identities that are referred to as *the sum and difference identities for sine and cosine* and *the double angle identities for sine and cosine.*

The Teaching Cycle:

Launch (Whole Class):

Introduce students to the goal of this task—which is to find identities that give the values of the sine and the cosine of an angle of rotation formed as the sum of two rotations. In the unit circle diagram, $sin(\alpha+\beta)$ is represented by the length of the dotted line segment RS and $cos(\alpha+\beta)$ is represented by the length of segment OS (or segment DR). Students will first focus on finding a way to represent the length of the dotted line segment RS in terms of the sine and cosine of angles α and β.

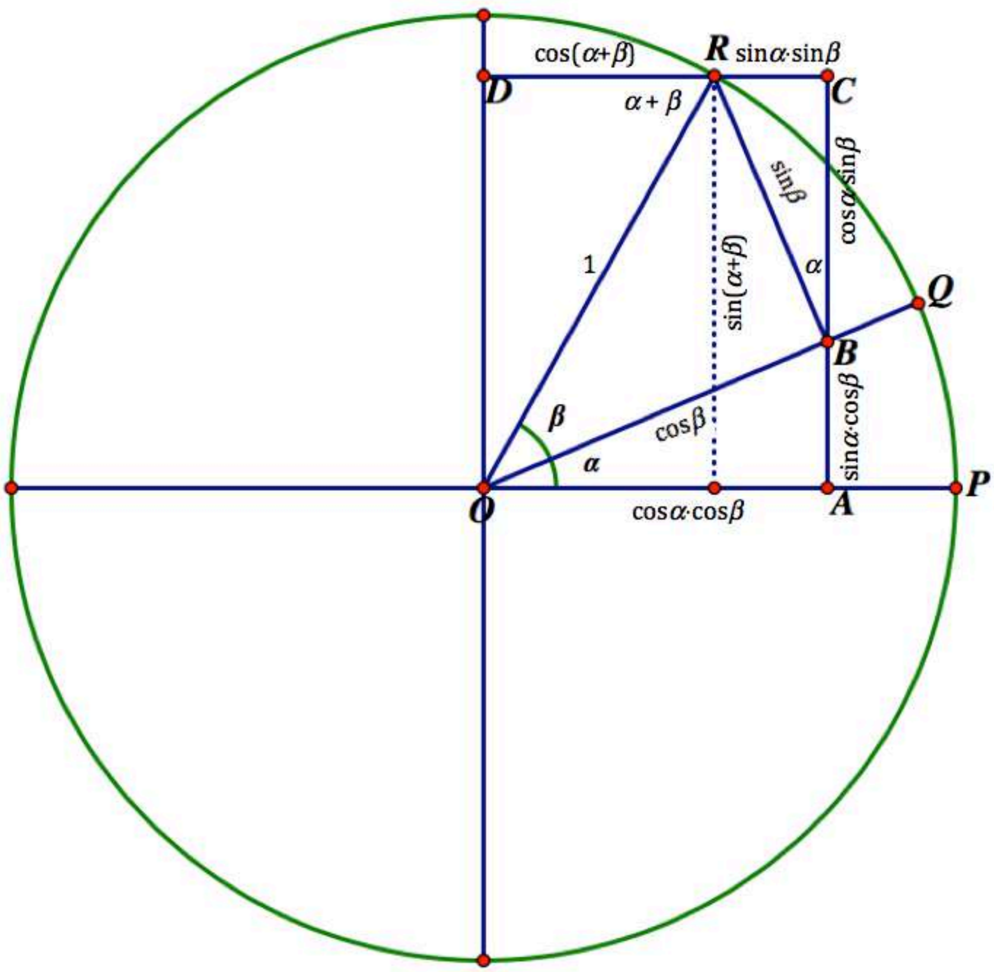
Question 1 allows students to begin reasoning about the relationships between these angles and sides. Give students a few minutes to answer question 1 individually, then discuss it as a class. As part of this discussion, press students to identify segments whose lengths represent $sin(\alpha)$ (that is, the segment through Q perpendicular to segment OP, which is not drawn in the diagram), and $sin(\beta)$ (which is segment RB, since segment OR, the hypotenuse of right triangle OBR, is also a radius of the unit circle, and therefore has length 1). The sum of these two segments is obviously not the same as the length of the dotted line segment RS. Therefore, $sin(\alpha + \beta)$ is not equal to $sin(\alpha)\ +\ sin(\beta)$.

Some students may mistakenly state that the segment whose length represents $sin(\alpha)$ is the vertical segment AB. Remind them that point B would have to lie on the unit circle for this to be the case. Whether this error in thinking comes up or not, at this time ask, "So, is there a way to determine the length of segment AB?" Inform students that the work of this part of the task is to find ways to represent the lengths of the line segments forming the rectangle $OACD$, and then to create identities for $sin(\alpha + \beta)$ and $cos(\alpha + \beta)$ from the relationships between the segments on the sides of this rectangle. Label the segments on the diagram whose lengths we have represented so far: $OR = 1$, $RB = sin\ \beta$, $RS = sin(\alpha + \beta)$, and $OS = DR = cos(\alpha + \beta)$. Suggest that students continue the exploration by first trying to determine the length of segment AB. Inform students that you have hint cards that can be distributed if they get stuck.

Explore (Small Group):

Observe as students try to find a trig relationship that would denote the length of segment AB. Students should note that $\sin(\alpha) = \frac{AB}{OB} \Rightarrow AB = \sin(\alpha) \cdot OB$. Students need to also notice that $OB = cos(\beta)$. Therefore, $AB = sin(\alpha) \cdot cos(\beta)$. Next, students will need to find an expression for the length of segment BC. Since this segment is a leg of right triangle BCR, it will be helpful to note that angle $CBR \cong$ angle AOB, and therefore, measures α radians. (These angles can be shown to be congruent by noting that segments OD and AC are parallel since they are opposite sides of a rectangle. These parallel sides have been cut by a transversal, line OQ, forming corresponding angles CBO and TOB. Each of these angles consists of a right angle and another pair of congruent angles measuring α. By similar reasoning, angles DRO and SOR are congruent and measure $\alpha+\beta$ radians.)

Watch for students who are making progress on representing the lengths of various line segments using trig expressions. Distribute the hint cards, as needed. A completed diagram should look like this:



Discuss (Whole Class):

Have students share the reasoning behind the labeling of the different segments of the diagram. Then use the diagram to complete the sum of angles identities:

$$\sin(\alpha+\beta)=\sin(\alpha)\cdot\cos(\beta)+\cos(\alpha)\cdot\sin(\beta)$$
$$\cos(\alpha+\beta)=\cos(\alpha)\cdot\cos(\beta)-\sin(\alpha)\cdot\sin(\beta)$$

Have students then present the algebraic derivations of the following identities, as asked for in questions 3, 5, 6 and 7.

$$\sin(\alpha-\beta)=\sin(\alpha)\cdot\cos(\beta)-\cos(a)\cdot\sin(\beta)$$
$$\cos(\alpha-\beta)=\cos(\alpha)\cdot\cos(\beta)+\sin(\alpha)\cdot\sin(\beta)$$
$$\sin(2\alpha)=2\sin\alpha\cdot\cos\alpha$$
$$\cos(2\alpha)=\cos^2\alpha-\sin^2\alpha$$

As a class, derive alternative forms of the double angle formula for $cos(2\alpha)$ by alternately substituting in one or the other of the following relationships, which can be derived from the Pythagorean identity $\sin^2\alpha+\cos^2\alpha=1$.

- Substituting $\cos^2\alpha=1-\sin^2\alpha$ yields: $\cos(2\alpha)=1-2\sin^2(\alpha)$
- Substituting $\sin^2\alpha=1-\cos^2\alpha$ yields: $\cos(2\alpha)=2\cos^2(a)-1$

Aligned Ready, Set, Go: Trigonometric Functions, Equations and Identities 7.7H

READY, SET, GO! Name Period Date

READY

Topic: Solving equations

Solve for x.

1. $2x - 1 = -2$
2. $6x - 3 = \sqrt{2}$
3. $x^2 - 3 = -2$
4. $4x^2 = 3$

5. How many solutions are there for #1 and #2?
6. How many solutions are there for #3 and #4?

7. $2(sinx) - 1 = -2$
8. $6(cosx) - 3 = \sqrt{2}$
9. $(sin)^2x - 3 = -2$
10. $4(cos)^2x = 3$

11. How many solutions are there for #7 and #8?
12. Problems 7 and 8 are very similar to problems 1 and 2. Why do 7 and 8 have more answers?
13. How many solutions are there for problems 9 and 10?
14. Problems 9 and 10 are very similar to problems 3 and 4. Why do 9 and 10 have more answers?

SET

Topic: Exploring the sum and difference identities

The sum and difference formulas are given below.

$sin(u + v) = (sinu)(cosv) + (cosu)(sinv)$	$cos(u + v) = (cosu)(cosv) - (sinu)(sinv)$
$sin(u - v) = (sinu)(cosv) - (cosu)(sinv)$	$cos(u - v) = (cosu)(cosv) + (sinu)(sinv)$

The sum and difference formulas can be used to find **exact** values of trigonometric functions involving sums or differences of special angles. The hard part is recognizing that an unfamiliar angle might be the sum or difference of an angle you know.

Write the following as the sum or difference of two angles for which you know the trig values.

15. $75°$
16. $345°$
17. $\frac{\pi}{12}$
18. $-\frac{13\pi}{12}$

Find the exact value of the following using the sum and difference formulas.

19. $sin75°$
20. $cos345°$
21. $cos\frac{\pi}{12}$
22. $sin\left(-\frac{13\pi}{12}\right)$

Find the exact value of the function given that $\boldsymbol{sin\ u = \frac{5}{13}\ and\ cosv = -\frac{3}{5}}$. (Both u and v are in Q. 2.)

23. $sin(u+v)$
24. $cos(u+v)$

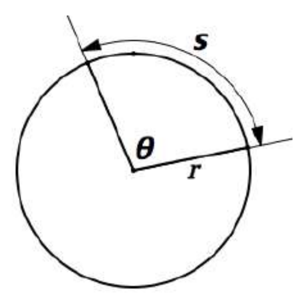
Find the exact value of the function given that $\boldsymbol{sin\ u = -\frac{7}{25}\ and\ cosv = -\frac{4}{5}}$. (u and v are in Q. 3.)

25. $sin(u-v)$
26. $cos(u-v)$

GO

Topic: Finding arc length

Recall the formula for arc length: $\boldsymbol{s = r\theta}$, where $\boldsymbol{\theta}$ is always in radians.
Write your answers with π in it. Then use your calculator to find the approximate length of the arc to 2 decimal places.



27. Find the length of an arc given that r = 10 in and $\theta = \frac{\pi}{4}$.
28. Find the arc length given r = 4 cm and $\theta = \frac{5\pi}{6}$.
29. Find the arc length given r = 72.0 ft and $\theta = \frac{\pi}{8}$.

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7.8H The Amazing Inverse Trig Function Race

A Solidify Understanding Task

To entertain themselves on weekends at the archeological dig, Javier has invented a game called “Find My Stake.” The game consists of drawing two cards, one from a deck of cards that Javier calls “The Angle Specification” cards, and the other from a deck Javier calls the “Location” cards. Based on these two clues, Veronica and Alyce race to locate the position of the stake. The friend who finds the correct location first, wins a prize. Alyce wonders why they need to have two clues. Veronica wonders if two clues will always be enough.

With a partner, play Javier’s game a few times using the two decks of cards that will be provided by your teacher. One of you will draw an “Angle Specification Card.” The other will draw a “Location Card.” See if you can determine the exact location of the stake that is described by the two clues given on the cards. Note that “Angle Specification” cards do not state an angle directly. Rather, they give information about the angle being specified, such as an inverse trig function statement or an equation to be solved. The “Location” cards give additional information to assist you in locating the stake, such as giving the x or y-coordinate of the stake (but not both); or giving r, the distance from the central tower; or perhaps telling the quadrant in which the stake is located.

The archelogical site is laid out using both a rectangular grid system and a circular grid system. In the rectangular grid system the horizontal axis represents distances east and west of the central tower, and the vertical axis represents distances north or south of the central tower, the same as on a conventional map. In the circular grid system, concentric circles surround the central tower at equally spaced intervals. Javier has provided both a rectangular grid map and a circular grid map of the archeological site for Veronica and Alyce to use while playing the game. Likewise, your teacher will provide you with both types of grids as you play the game.

Playing the Game

With your partner, play the game at least three times as described above. For each time you play the game, do the following:

- Record the two clues you draw, one from each deck
- Show all work, including calcuations, that you do in an attempt to locate the stake
- Choose a rectangular grid or circular grid on which you will record the location of the stake—if you cannot locate the stake exactly, show all possible locations of the stake on the grid; if the clues provide contradictory information, state that a location is impossible to determine
- If possible, determine the location of the stake on both the rectangular and circular grids

1. Recall that Veronica wondered if two clues would be enough to locate the stake. After playing the game a few times, what do you think?

Analyzing the Game

Examine the clues given to you in the two decks of cards, and then do the following:

- Select a pair of cards that would determine a specific location for the stake—record the clues on the cards and explain why they determine a single, unique location
- Select another pair of cards that would suggest the stake can be located in more than one location—record the clues on the cards and explain why the location of the stake is not uniquely specified
- Select a third pair of cards that give contradictory information—record the clues on the cards and explain the conflict

Repeat these steps a few times until you can answer the following question.

2. *In general, what types of combinations of clues specify a unique location?*

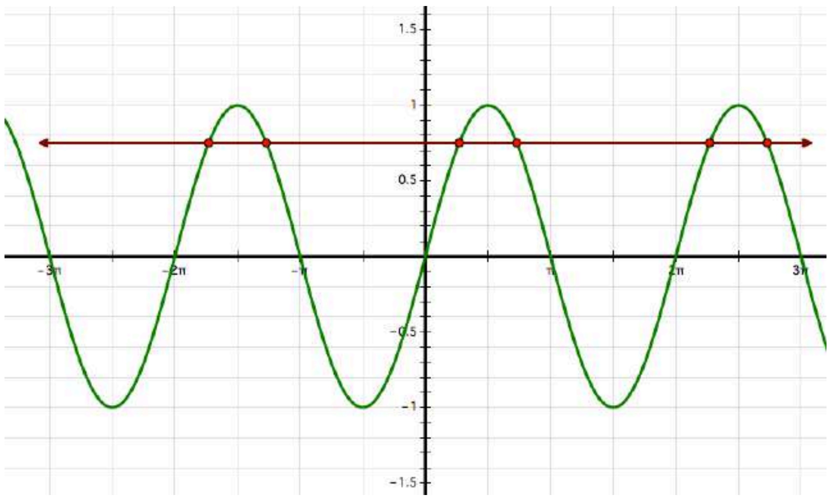
Explaining the Game

For each of the "Angle Specification" cards you had to answer the question, "What angle could fit this given information?" Perhaps you thought about the unit circle or used a calculator to answer this question. For angles of rotation, there are many answers to this question. Therefore, this question—by itself—does not define an inverse trig <u>function</u>.

Suppose you draw this clue from the set of "Angle Specification" cards:

$$\sin \theta = 0.75$$

Using your calculator, $\sin^{-1}(0.75) = 0.848$ radians. However, the following graph indicates other values of θ for which $\sin \theta = 0.75$.



3. Without tracing the graph or using any other calculator analysis tools, use the fact that $\sin^{-1}(0.75) = 0.848$ radians to find at least three other angles θ where $\sin \theta = 0.75$. (Each of these points shows up as a point of intersection between the sine curve and the line $y = 0.75$ in the graph shown above.)

Your calculator has been programmed to use the following definition for the inverse sine function, so that each time we find ***sin⁻¹*** of a number, we will get a unique solution.

***<u>Definition of the inverse sine function</u>*: $y = \sin^{-1} x$ means, "find the angle y,**

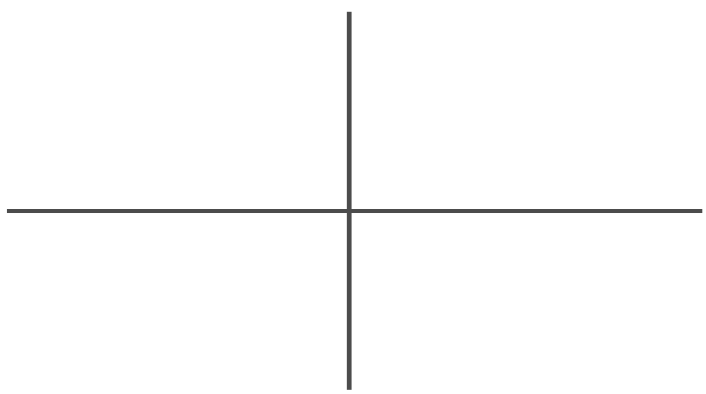
***on the interval* $-\pi/2 \le y \le \pi/2$, such that $\sin y = x$."**

4. Based on the graph of the sine function, explain why defining the inverse trig function in this way guarantees that it will have a single, unique output.

5. Based on this definition, what is the domain of the inverse trig function?

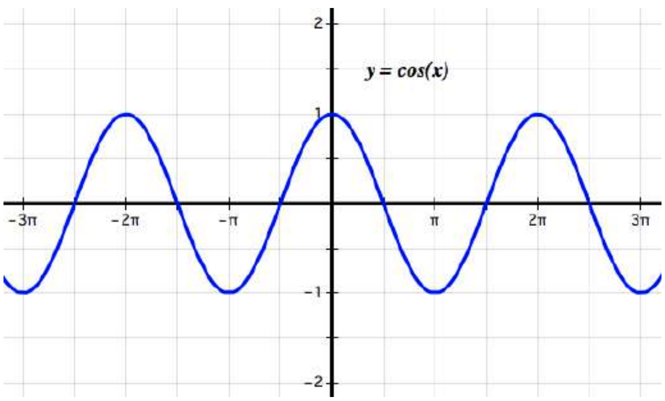
6. Based on this definition, what is the range of the inverse trig function?

7. Sketch a graph of the inverse sine function.

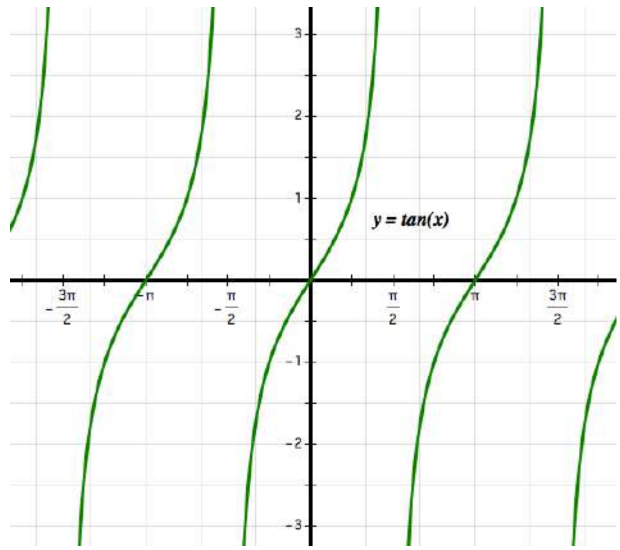


8. Suppose you draw this clue from the set of "Angle Specification" cards: sin $\theta = -½$. What is the exact answer to this inverse sine expression: $\sin^{-1}(-½)$?

Examine the graphs of the cosine function and the tangent function given below. How would you restrict the domains of these trig functions so that the inverse cosine function and the inverse tangent function can be constructed?



9.



9. Complete the definitions of the inverse cosine function and the inverse tangent function below. State the domain and range of each function, and sketch its graph.

Definition of the inverse cosine function:

Domain: Range:

Graph:

Definition of the inverse tangent function:

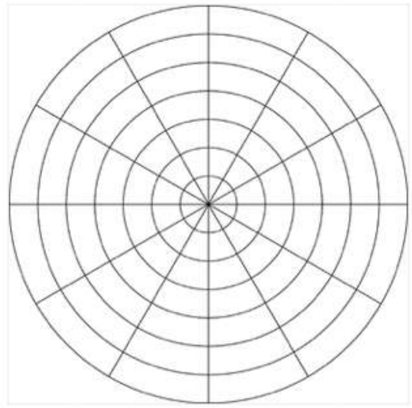
Domain: Range:

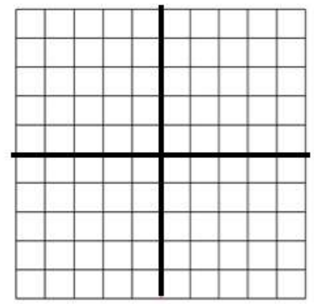
Graph:

Game 1

Location clue:

Angle clue:

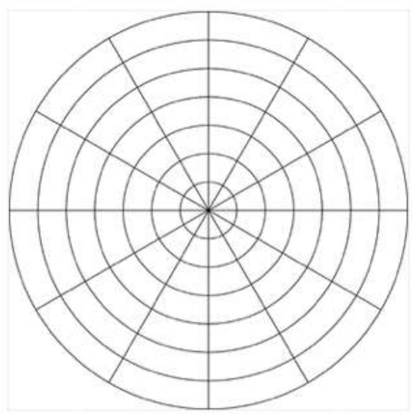


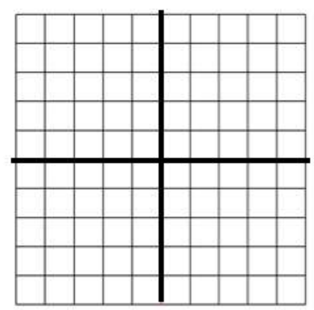


Game 2

Location clue:

Angle clue:

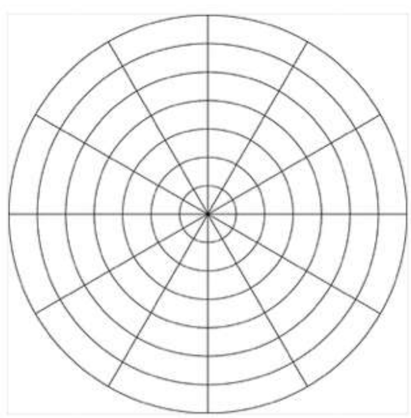


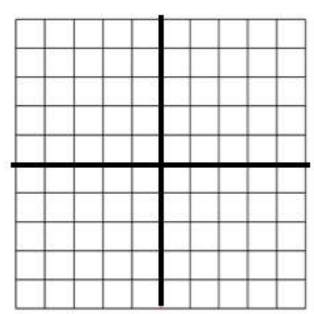


Game 3

Location clue:

Angle clue:





7.8H The Amazing Trig Function Race – Teacher Notes

A Solidify Understanding Task

Purpose: This task extends students thinking about inverse trig functions and examines the graph of the inverse sine, inverse cosine and inverse tangent functions. These functions are constructed by first considering intervals on the graphs of the sine, cosine and tangent functions that are always increasing or decreasing. Students examine the difference between the question, "What angles are specified by this trig statement?" and the question answered by an inverse trig function, "What angle *in the interval used to define this inverse trig function* is specified by this trig statement?" In this task, students may also use trig identities, as well as inverse trig functions, to solve trig equations.

Core Standards Focus:
F.TF.6 (+) Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.

F.TF.7 (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.★

Standards for Mathematical Practice:
SMP 2 – Reason abstractly and quantitatively
Vocabulary: Students will need to understand the *restricted domains* that are necessary when defining the inverse trig functions

Note to teacher #1: This task will take two days. On the first day, students should play several rounds of the game so they will have a number of opportunities to encounter all of the variety of issues raised when solving trigonometric equations, using inverse trig functions, and locating points on polar and coordinate grids. On the second day, students should analyze and explain the game, leading to formal definitions of the inverse trig functions.

Note to teacher #2: The cards for the game appear at the end of these teacher notes. The recording grids for each game are attached as the last page of the student task. Make enough sets of the game cards for each 3 or 4 pairs of students in your class. Students will play the game in pairs, but multiple pairs of students can share the same deck. Make sure that students return their clue cards to the decks before selecting another set of clues. Run the angle specification cards and location cards on two different colors so they don't get mixed together. You may also want to selectively choose which cards to include in specific decks, in order to increase or lower the level of expectation for a particular group of students.

An answer key for the questions in the task can also be found at the end of these teacher notes. It is recommended that you work through the task yourself before consulting the answer key to develop a better sense of how your students might engage in the task.

Masters for the locations cards and "angle specification cards" that are used to play the game described in the task are located at the end of the teacher notes.

The Teaching Cycle:

Launch (Whole Class):

Remind students of their work on the task *High Tide*, where they were asked to answer "inverse trig" questions—that is, instead of being given the time and asked to find the height of the tide, they were sometimes given the height of the tide and asked to find the time it was at this position. In that task they found that the calculator function $\sin^{-1}(x)$ gave them results that were limited to a specific interval, and sometimes didn't answer the specific question they were being asked. In this task students will take a closer look at how inverse trig functions are constructed.

Read paragraph 1 of the task and remind students of their work with the Javier, Alyce and Veronica at the archeological dig discussed in previous tasks.

Read paragraph 2, and then distribute the two decks of cards. Inform students that since each game only requires one card from each deck, they do not need a set of cards for each pair of students. A larger group of students can work from the same deck, but students should play the actual game as partners. For each game, one of the partners will select a card from the "Angle Specification" deck, and the other partner will select a card from the "Location" deck. Before they play again they should return the cards to the deck so they are available for other partnerships to draw. As you distribute the cards, let students browse through the two sets of cards so they get familiar with the types of clues contained in each set, as described at the end of paragraph 2.

As you read paragraph 3, distribute a playing mat to each pair of students. Point out that on one side of the mat they are given rectangular grids on which to locate their stakes. On the other side of the mat they are given circular grids. For each game, the partnership should decide which grid is most useful to use in conjunction with the clues they draw.

Once the deck of cards and playing mats have been distributed, review how students are to play the game and what they are expected to record, as outlined in the section "Playing the Game." Then let them draw their first pair of cards and try to locate their first stake, based on the clues drawn.

Explore (Small Group):
Some partnerships will draw cards that define a unique location of a stake. They should plot this unique position on either a rectangular grid or a circular grid and label it as game 1. Others will draw cards that do not determine a unique position, but for which several positions are possible. They should mark all possible positions on either a rectangular grid or a circular grid and label it as game 1. Some partnerships will draw clues for which no position will satisfy both clues. They should record that no location is possible. Once game 1 is finished partners should return their clue cards to the appropriate deck and select another set of clues.

Note that all partnerships will have to do some algebraic work based on their "Angle Selection Card." They may need to solve an equation or evaluate an inverse trig function using a calculator or the unit circle. Some equations require students to use trig identities as part of the solution process. This work is the "practice" part of the task, so watch how adept students are at this work, and select particularly difficult work for the whole class discussion.

After partnerships have played the game three times, have them move to the "Analyzing the Game" portion of the task. Note that the "Explaining the Game" section of the task is intended to guide the whole class discussion, however, some students may begin working on this last section of the task before your bring the whole class together.

Discuss (Whole Class):
Begin the whole class discussion by having a partnership share a pair of clues that determined a unique position for the stake. Then have a partnership share a pair of clues that determined a set of different possible positions for the stake. Finally, have a partnership share a pair of clues that were impossible to use for determining the position of a stake.

Once these specific examples have been presented, move to the question *"In general, what types of combinations of clues specify a unique location?"* List the criteria as mentioned by the students, such as, a specific x or y coordinate is needed if the angle is specified by a trig equation; or if both the sine and cosine of the angle is specified, then an r value can be used to determine a unique position, but some x or y values may conflict with the angle information. Students need not identify all possibilities—the point of this discussion is to identify the issues involved when solving trig equations and when working with the inverse trig functions. Once these issues are starting to emerge, move to a discussion of the last part of the task, "Explaining the Game."

Discuss the "Explaining the Game" section of the task in several short *launch-explore-discuss* cycles. For example, read question 3, give students a couple of minutes to answer this question individually or with a partner, and then have someone share their work with the whole class. Make sure that on question 3 a strategy emerges for naming both sequences of solutions: $0.848 \pm 2\pi n$ and $2.294 \pm 2\pi n$.

Next read questions 4-7 and give students a few minutes to determine the domain and range of the inverse sine function and sketch its graph. The discussion should highlight the following: The domain is $-1 \leq x \leq 1$ and the range is $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, as stated in the definition. Relate this domain and range to the corresponding restricted interval of the sine function from which the inverse sine function is constructed. Also relate the graph of the inverse sine function to this restricted interval of the sine function.

Next, have students refer to the unit circle and give them a couple of minutes to answer question 8. If students respond that $sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ that is a good indicator of their understanding of the definition of the inverse sine function.

Finally, have students propose restricted intervals of the cosine and tangent functions from which the inverse cosine and inverse tangent functions can be constructed. The convention for the cosine function is the interval $[0, \pi]$, and an obvious interval for the tangent is the portion of the graph on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Once these intervals have been selected, have students write the definitions of the inverse cosine and inverse tangent functions.

If there were particularly difficult equations for students to solve, you may wish to discuss them at this time.

Another possible piece of mathematics to discuss relates to locating a stake on both the rectangular grid (i.e,. giving the *x*- and *y*-coordinates of the stake) and the polar grid (i.e., giving the *r*- and θ-coordinates of the stake). Have students describe strategies for converting from polar to rectangular coordinates, and vice versa. This work is also discussed in the next task, so you may want to introduce the ideas using the next task and then return to the game in this task, and play it again with an emphasis on locating stakes on each grid.

Aligned Ready, Set, Go: Trigonometric Functions, Equations and Identities 7.8H

Location Card $r = 10$	**Location Card** $x > 0$ and $y < 0$	**Location Card** $x = -20$
Location Card $r = 20$ and $y < 0$	**Location Card** $x < 0$ and $y > 0$	**Location Card** $r = 20$
Location Card $y = 20$	**Location Card** $r = 30$ and $x < 0$	**Location Card** $x < 0$ and $y < 0$
Location Card $r = 30$	**Location Card** $x = 30$	**Location Card** $r = 10$ and $y > 0$
Location Card $x > 0$ and $y > 0$	**Location Card** $y = -10$	**Location Card** $r = 40$

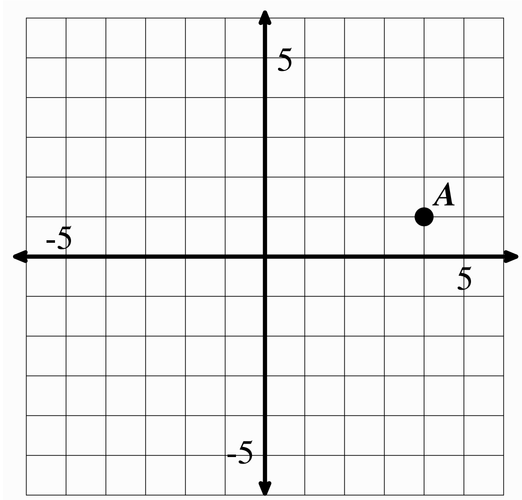
Angle Specification Card $2\sin\theta = \sqrt{3}$	**Angle Specification Card** $\cos^2\theta = \frac{3}{4}$	**Angle Specification Card** $\theta = \frac{2\pi}{3}$
Angle Specification Card $\theta = sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$	**Angle Specification Card** $\tan^2\theta = 3$	**Angle Specification Card** $\theta = \frac{5\pi}{4}$
Angle Specification Card $\tan^{-1}1 = \theta$	**Angle Specification Card** $\sin\theta = \frac{\sqrt{3}}{2}$ *and* $\cos\theta = -\frac{1}{2}$	**Angle Specification Card** $\sin^2\theta = \frac{3}{4}$ *and* $\cos\theta = \frac{1}{2}$
Angle Specification Card **lies on the line *y* = 2*x***	**Angle Specification Card** **lies on the line *y* = -3*x***	**Angle Specification Card** **lies on the line *y* = ½ *x***
Angle Specification Card $2\cos^2\theta - \sqrt{3}\cos\theta = 0$	**Angle Specification Card** $\sin^2\theta - \sin\theta - 2 = 0$	**Angle Specification Card** $\cos\theta - \cos^2\theta = \sin^2\theta$

READY, SET, GO! Name Period Date

READY

Topic: Recalling transformations using geometric notation

Transform point A as indicated below.

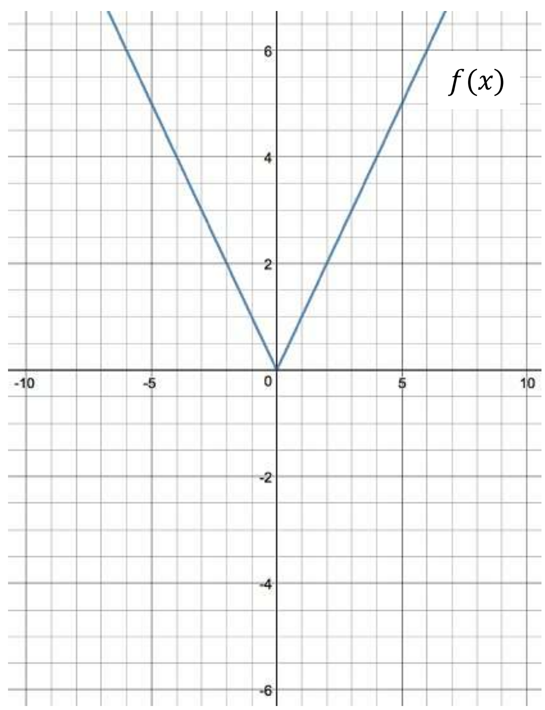


1. a. Apply the rule $(x, y) \rightarrow (x-2, y-5)$ to point A. Label as A'

   b. Apply the rule $(x, y) \rightarrow (x-1, y+3)$ to point A'. Label as A''

   c. Apply the rule $(x, y) \rightarrow (-2x, y)$ to point A'' and Label A'''

Transform the given graph as indicated below.



2. a. Apply the rule $(x, y) \rightarrow (x-3, y)$ to $f(x)$. Label $f'(x)$.

   b. Apply the rule $(x, y) \rightarrow (x, y-5)$ to $f(x)$. Label $f''(x)$.

   c. Apply the rule $(x, y) \rightarrow \left(x, \frac{1}{2}y\right)$ to $f(x)$. Label $f'''(x)$.

SET

Topic: Finding a specific angle of rotation using inverse trig functions

Use the given information to find the missing angle $(0 \leq \theta \leq 2\pi)$.
Round answers to thousandths place (3 decimal places).

3. $\cos\theta = 0.9848$; $\sin\theta > 0$

4. $\sin\theta = 0.9925$; $\tan\theta < 0$

5. $\cos\theta = 0.0872$; θ *is in Quadrant IV*

6. $\tan\theta = 0.3839$; $\cos\theta < 0$

7. $\cos\theta = 0$; $\sin\theta > 0$

8. $\sin\theta = -0.1908$; $\tan\theta > 0$

9. $\tan\theta = -0.4663$; $\sin\theta > 0$

10. $\tan\theta = -0.4663$; $\cos\theta > 0$

11. $\tan\theta = -1$; $\sin\theta > 0$

12. $\sin\theta = -1$

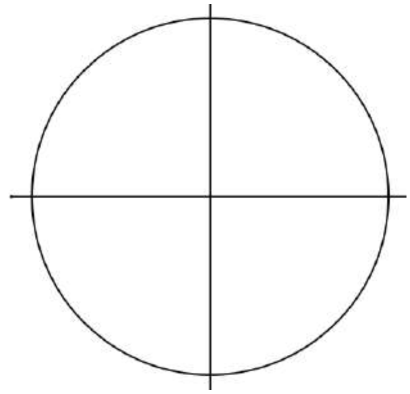
13. Explain why #12 needed only 1 clue to determine a unique value for θ, and #3 – 11 required at least 2 clues.

GO

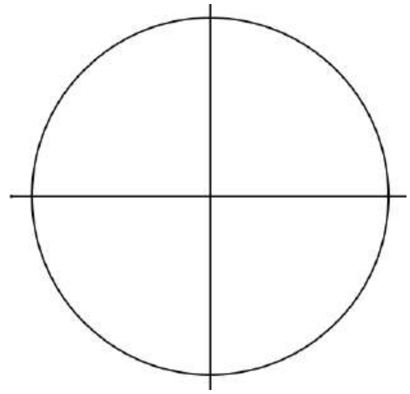
Topic: Calculating the area of a sector

Find the area of a sector of a circle having radius r and central angle θ. $\left(A = \frac{1}{2}r^2\theta\right)$ Make a sketch and shade in the sector. Round answers to 2 decimals.

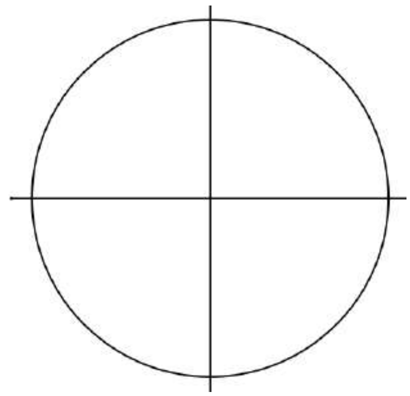
14. $r = 15\,m;\ \theta = \frac{2\pi}{5}\ radians$



15. $r = 29.2\,m;\ \theta = \frac{5\pi}{6}\ radians$



16. $r = 32\,mm;\ \theta = \frac{5\pi}{4}\ radians$



7.9H More Hidden Identities

A Practice Understanding Task

Note: Because trig functions are periodic, trig equations generally have multiple solutions. In this task you should find all solutions to the trig equations by representing a sequence of solutions using the form $answer \pm k\pi n$, where $k\pi$ represents the interval between successive solutions.

In *Hidden Identities,* Javier observed that he might need to change the form of the trig expressions in a trig equation before he could use an inverse trig function to solve the equation. Changing the form of a trig expression involves looking for trig identities that apply to the expression. Sometimes you have to manipulate the expression algebraically before you can identify a trig identity that might apply.

Here is a synopsis of how Javier would have solved a problem in *Hidden Identities.* You can use his strategy to solve the remaining problems in this task.

1. Solve: $2\sin(x)\cos(x) + \frac{\sqrt{3}}{2} = \sqrt{3}$

Idea #1: Do you see a trig identity? If so, write the equation in an equivalent form using the identity. If not, can you manipulate the trig equation algebraically?

a. Which approach do you think Javier would need to take for his first step to solve this problem: use an identity or manipulate the equation algebraically?

b. Apply your thought:

Idea #2: Repeat idea #1 until you have isolated the trig function on one side of the equation.

c. Apply idea #2:

Idea #3: Apply the inverse trig function to both sides of the equation, which will give an equation of the form: *an expression* = *an angle;* solve this equation for the variable.

d. Apply idea #3:

Idea #4: This process finds one of an infinite number of solutions. Since trig functions are periodic, adding or subtracting multiple increments of the period will generate new solutions. Write this infinite set of solutions in the form $answer \pm k\pi n$, where $k\pi$ represents the interval between successive solutions.

e. Apply idea #4:

Idea #5: Since the inverse trig function used in #3 *is a function*, we only get one unique value as a solution to the inverse trig expression, while there may be another solution in a different quadrant on the unit circle. Find this alternate solution, and the corresponding infinite set of solutions to the equation.

f. Apply idea #5:

Hints for solving trig equations: Graphs of trig functions and the unit circle can be used as tools to help reason about the solutions to a trig equation.

g. Illustrate how a graph and the unit circle can help you find the solutions to this equation:

Solve the following trig equations by applying the ideas and hints described above.

2. $2\cos(3x) = \sqrt{2}$

3. $\sin^2(x) - \sin(x) - 2 = 0$

4. $2\cos^2(x) - \sqrt{3}\cos(x) = 0$

5. $(\sin(x) + \cos(x))^2 = \frac{3}{2}$

7.9H More Hidden Identities – Teacher Notes

A Practice Understanding Task

In this task, students draw upon the identities discovered in previous tasks to help them change the form of trigonometric expressions into simpler expressions until the solutions of the equation can be found. The dilemma of finding all of the solutions to an equation when using an inverse trig function to solve a trig equation surfaces again, and students will learn how to represent all possible real number solutions to a trig equation.

Core Standards Focus:

F.TF.7 (+) Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.*

Standards for Mathematical Practice:

SMP 7 – Look for and make use of structure

The Teaching Cycle:

Launch (Whole Class):

Post equation #1 on the board and have students work in pairs to solve this problem using the list of five ideas and the hints about using graphs and the unit circle as tools for solving equations. As students are working through this list of ideas and hints, monitor their small group discussions and stop the class to hold a whole class discussion if several students are struggling with a particular idea. Make sure that students understand the implications of each hint before assigning students to work on the remainder of the task.

Explore (Small Group):

As students are working on trig equations 2-5, respond to any questions they may have by referring them back to the list of five ideas for solving trig equations. It is important that they see the value in thinking through a general procedure for solving such equations, as well as being flexible enough to revise or adapt a procedure to handle new and unique cases. Select students to present their equation-solving processes on questions 2-5 who have shown enough detail in their work to make their thinking clear to their peers.

Discuss (Whole Class):

Have students present their solution processes for each trig equation 2-5. As they do so, press them to connect their work to the five ideas for solving trig equations presented in the task. Note that each problem has been selected to illustrate different ways of applying the five ideas, so it is good to focus on the similarities and differences between each problem. Therefore, it would be good to retain all of the solution processes for problems 2-4 on the board at the same time, and to focus some of the discussion on comparing strategies.

Exit ticket for students: *Solve the following equation for* θ on the interval $-\infty < \theta < \infty$:

$$\cos^2(2\theta) = \frac{1}{2}$$

The exit ticket is intended to be formative assessment for both the teacher and student and therefore, should be completed by students independently. Formative assessment is most effective when students are given specific feedback so that they understand their progress relative to the standard. Discussion of the previous day's exit slip is a great warm-up for a lesson.

Instructional Supports:

Procedures as generalized strategies: This task approaches the procedural work of solving trigonometric equations as a collection of generalized ideas to guide thinking. Procedures are more than a set of memorized steps. They are methodologies that can be adapted flexibly to fit individual cases. Rather than just working through a set of examples, this tasks provides students with a set of ideas that can be adapted, modified, or revised.

Aligned Ready, Set, Go: Trigonometric Functions, Equations and Identities 7.9H

READY, SET, GO! Name Period Date

READY

Topic: Examining forms of linear and quadratic functions

The different forms of linear and quadratic functions are listed below. Explain how the structure of each form gives you information about the graph of the function.

Linear:

1. Standard form: $ax + by = c$

2. Slope-intercept form: $y = mx + b$

3. Point-slope form: $y = m(x - x_1) + y_1$

Quadratic:

4. Standard form: $y = ax^2 + bx + c$

5. Factored form: $y = a(x - r_1)(x - r_2)$

6. Vertex form: $y = a(x - h)^2 + k$

SET

Topic: Solving trigonometric equations

Solve the following trig equations. Write your answer(s) in the form $\pm k\pi n$. where $k\pi$ represents the interval between successive solutions.

7. $5cosx + \sqrt{2} = \sqrt{2}$

8. $2sinx + \sqrt{2} = 0$

9. $tanx + 2\sqrt{3} = 3\sqrt{3}$

10. $sin^2x = 3cos^2x$

11. $2 - sin^2x = 1 + cosx$

12. $2sin^2x + 3sinx + 1 = 0$

13. $2sin^2(2x) = 1$

14. $2cos^2x - cosx = 1$

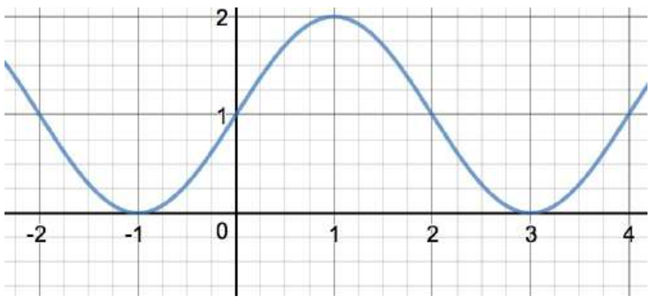
15. $2cos^2x - 5cosx + 2 = 0$

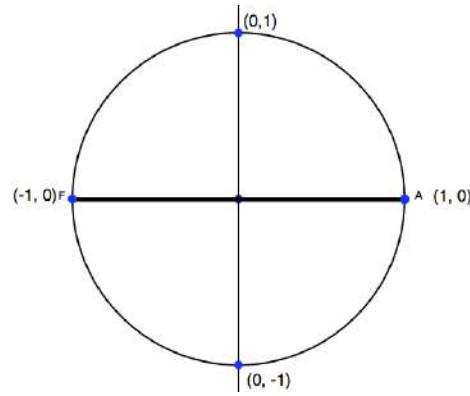
GO

Topic: Interpreting graphs

Use the graph to find <u>all</u> of the values for x when y=0 for the given equation. Write your answer(s) in the form $\pm k\pi n$. where $k\pi$ represents the interval between successive solutions.

15. $y = sin\left(\frac{\pi x}{2}\right) + 1$

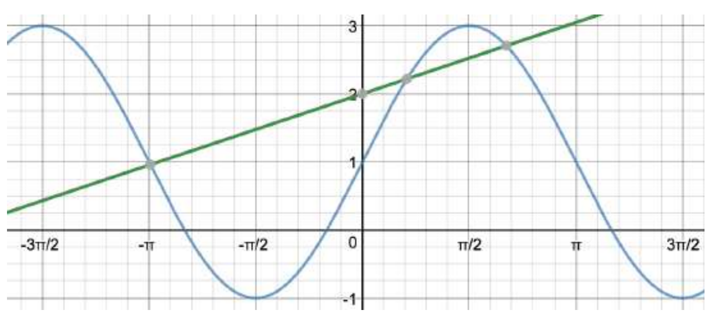




16. Use the unit circle to explain the solutions you found in problem 15.

Use the graph to approximate the points of intersection of the graphs of y_1 and y_2.

17. $y_1 = 2sinx + 1$
$y_2 = \frac{1}{3}x + 2$



18. The scale on the x-axis in the graph of problem 15 is 1. The scale in the graph of problem 17 is π/2. Yet the units on both axes is radians.

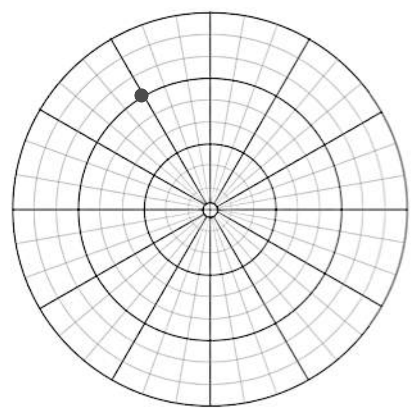
A.) Label the graph in #15 with the approximate location of $\frac{\pi}{2}$ and π.

B.) Label the graph in #17 with the approximate location of 1, 2, 3, and 4.

7.10H Polar Planes

A Develop Understanding Task

Alyce, Javier and Veronica have two different ways of recording the location of artifacts at the archeological dig: one way is to use **rectangular coordinates** (x, y) and the other is to use **polar coordinates** (r, θ). The three friends know a lot about plotting points and graphing functions on a rectangular coordinate grid, and they are wondering if they can sketch graphs on a polar grid. They have found some polar graph paper in the archeological supplies, and are trying to make sense of it. They have learned that angles are measured with the initial ray pointing horizontally to the right (the positive horizontal axis), and positive angles are measured counterclockwise.



Javier thinks the location of the point plotted on the polar grid at the right is given by the polar coordinates (6, 120°).

Alyce thinks the location of the point is (6, -240°).

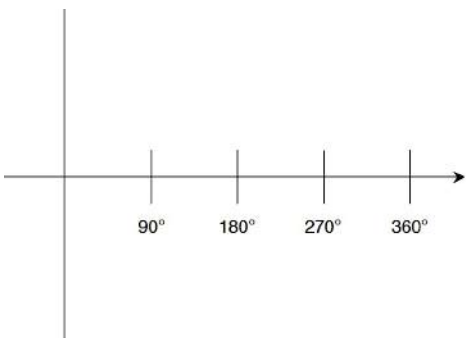
Veronica thinks the location of the point is (-6, 300°).

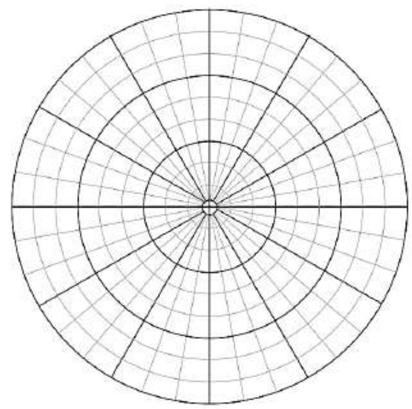
1. What do you think? Who has named the location of the point correctly? Explain why?

2. What are the rectangular coordinates of the plotted point?

Javier knows how to sketch the graph of $y = 6\sin(2\theta)$ on a rectangular grid, and wonders what the graph of $r = 6\sin(2\theta)$ might look like on a polar grid. When graphing trigonometric functions on a rectangular grid, Javier looks for key points, like maximums, minimums and x- and y-intercepts. He wonders how these strategies might work on a polar grid. Of course, he knows he can always create a table of values, plot a few consecutive points, and then connect them with a curve if he needs to.

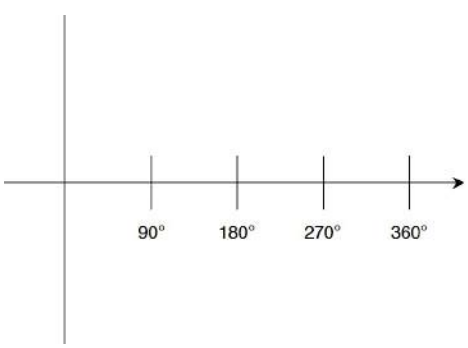
3. Sketch a graph of $y = 6\sin(2\theta)$ on the rectangular grid below and a graph of $r = 6\sin(2\theta)$ on the polar grid.

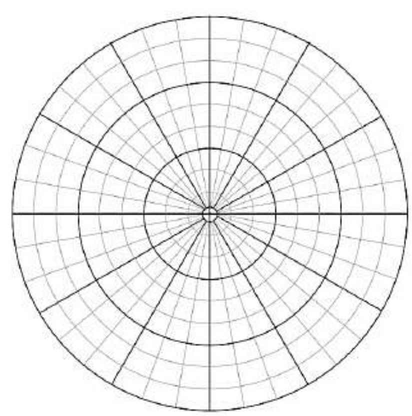




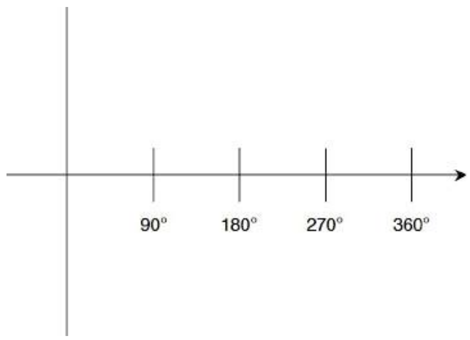
Javier wants to continue to compare the graphs of similar curves written in rectangular and polar form.

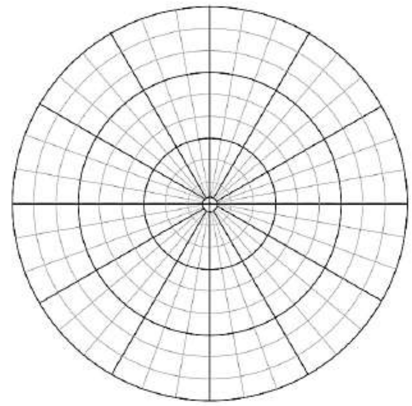
4. Graph $y = 6\sin(3\theta)$ and $r = 6\sin(3\theta)$ on the rectangular and polar grids.





5. Graph $y = 2 + 4\sin(\theta)$ and $r = 2 + 4\sin(\theta)$ on the rectangular and polar grids.





6. What strategies seem to work when graphing curves on a polar grid, so you don't have to plot a lot of individual points? Describe as many strategies as you can.

7.10H Polar Planes – Teacher Notes

A Develop Understanding Task

Purpose: Students have been using polar coordinates in their work with laying out stakes on a circular grid for the archeological site. In the task, work with polar coordinates and polar grids are formalized. Graphing polar curves, $r = f(\theta)$, serve as a context for familiarizing students with plotting points on a polar grid.

Core Standards Focus:

N.CN.4. (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

N.CN.5. (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. For example, $(-1 + \sqrt{3}\, i)^3 = 8$ because $(-1 + \sqrt{3}\, i)$ has modulus 2 and argument 120°.

MVP Honors Standards:

- Define and use polar coordinates and relate them to Cartesian coordinates.
- Represent complex numbers in rectangular and polar form, and convert between rectangular and polar form.
- Translate equations in Cartesian coordinates into polar coordinates and graph them in the polar coordinate plane.
- Multiply complex numbers in polar form and use DeMoivre's Theorem to find roots of complex numbers.

Related Standards: N.CN.1, N.CN.2, N.CN.3

Standards for Mathematical Practice:
SMP 7 – Look for and make use of structure

Vocabulary: In this task students will distinguish between *rectangular coordinates* (ordered-pairs of the form (x, y) plotted on a rectangular/Cartesian grid) and *polar coordinates* (ordered-pairs of the form (r, θ) plotted on a polar grid) as two different ways of locating points in a plane, and they will examine features of the graphs of *polar curves* plotted on a polar grid.

The Teaching Cycle:
Launch (Whole Class):
Use questions 1 and 2 of the task to launch the idea that while a point in a plane has only one ordered-pair that represents its position in rectangular coordinates (x, y), it can have many different ordered-pairs (r, θ) that represent its position in polar coordinates; that is, Alyce, Javier and Veronica have all named the point correctly. Make sure students understand Veronica's use of a negative value for r. Review Javier's strategies for plotting trigonometric functions on a rectangular grid, and then let students explore how these strategies play out on a polar grid for related functions $r = f(\theta)$.

Explore (Small Group):
To help students visualize the effects of a curve drawn on a rectangular grid vs. a polar grid, it may be helpful to have students visualize the result of tracing a point that is oscillating up and down (such as the point of a pen or whiteboard marker) as a paper moves horizontally beneath the point, versus a paper that rotates beneath the point.

Watch for students who plot the x-intercepts on the rectangular grid as points at the pole of the polar grid, and students who plot portions of the graph that go below the x-axis on the rectangular grid as having negative r-values on the polar grid.

Discuss (Whole Class):

Have students share their strategies for plotting polar curves, including how to plot positive values of *r*, negative values of *r*, and 0 values for *r*. Also ask students to explain the number of petals or other interesting shapes of the graphs that are produced.

Note to teachers: To address the MVP Honors standard "Translate equations in Cartesian coordinates into polar coordinates and graph them in the polar plane," inform students that sometimes it is useful to write equations in rectangular form in polar form (or vise versa). The work they have been doing to convert coordinates can also be used to convert expressions. For example, since

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ r = \sqrt{x^2+y^2}\ \ and\ \ \theta = \tan^{-1}\left(\frac{x}{y}\right)$$

we can substitute equivalent expressions for *x, y, r* or *θ*. Have students graph y = 2x + 1 on a rectangular grid, then find the corresponding polar form of this rectangular equation and plot it on a polar grid. (Note, they will need to solve the resulting corresponding equation $r\sin\theta = 2r\cos\theta + 1$ for *r*, (i.e., $r = \frac{1}{\sin\theta - 2\cos\theta}$), in order to graph the equation in polar form. They might use technology or calculate a table of points to plot this graph. They should see that these two equations produce identical graphs regardless of the chosen coordinate grid.

Aligned Ready, Set, Go: Trigonometric Functions, Equations and Identities 7.10H

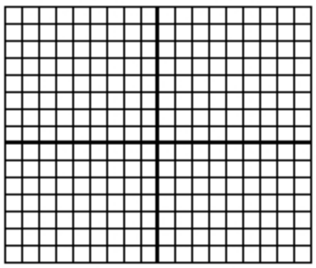
READY, SET, GO! Name Period Date

READY

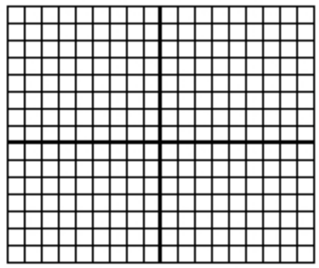
Topic: Applying the distance formula

Find the distance from the origin (0, 0) to the given point in the rectangular plane.

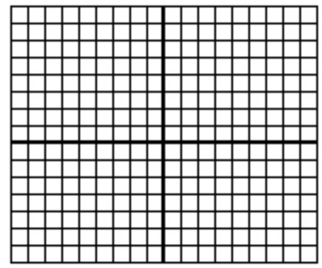
1. A (8, 6)



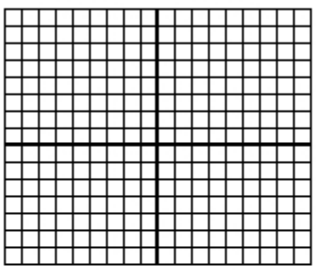
2. P (-5, -6)



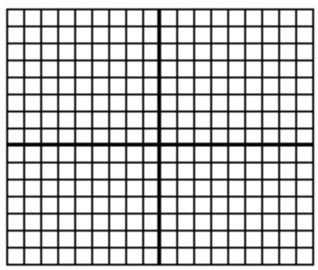
3. F (-7, 7)



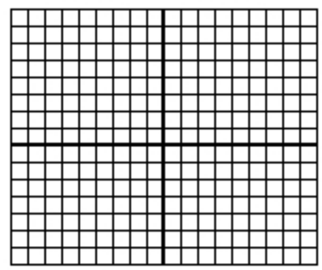
4. R (3, -4)



5. $G\left(\sqrt{3}, 1\right)$



6. $Q\left(3, \sqrt{7}\right)$

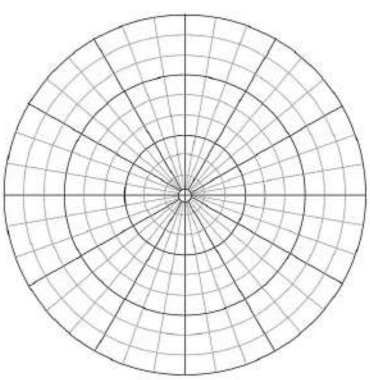


SET

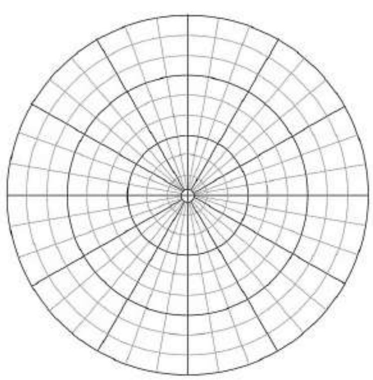
Topic: Graphing polar coordinates

Plot the given point (r, θ) in the Polar Coordinate System.

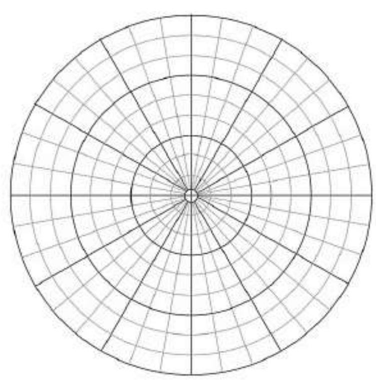
7. $(\boldsymbol{r}, \boldsymbol{\theta}) = \left(\mathbf{5}, \frac{2\pi}{3}\right)$



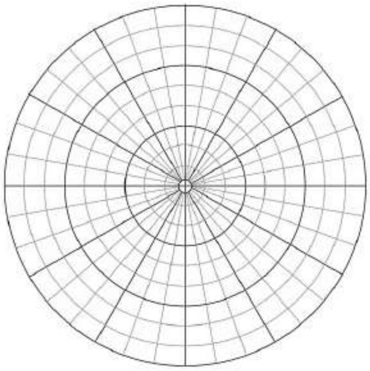
8. $(\boldsymbol{r}, \boldsymbol{\theta}) = \left(\mathbf{3}, \frac{7\pi}{6}\right)$



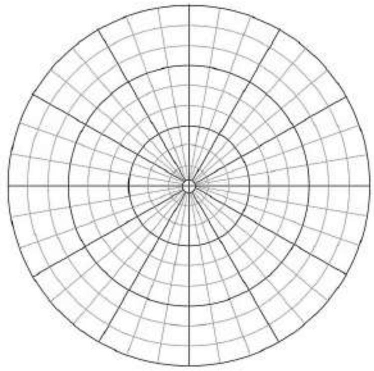
9. $(\boldsymbol{r}, \boldsymbol{\theta}) = \left(\mathbf{8}, \frac{5\pi}{3}\right)$



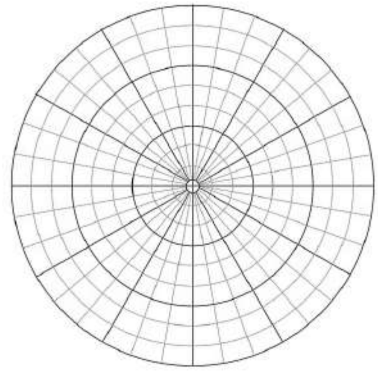
10. $(\boldsymbol{r}, \boldsymbol{\theta}) = \left(\mathbf{9},\ \frac{\boldsymbol{\pi}}{\mathbf{3}}\right)$



11. $(\boldsymbol{r}, \boldsymbol{\theta}) = \left(\mathbf{7},\ \frac{\mathbf{5}\boldsymbol{\pi}}{\mathbf{6}}\right)$

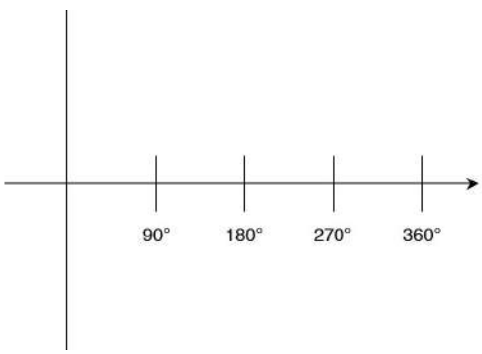


12. $(\boldsymbol{r}, \boldsymbol{\theta}) = \left(\mathbf{-6},\ \frac{\boldsymbol{\pi}}{\mathbf{3}}\right)$

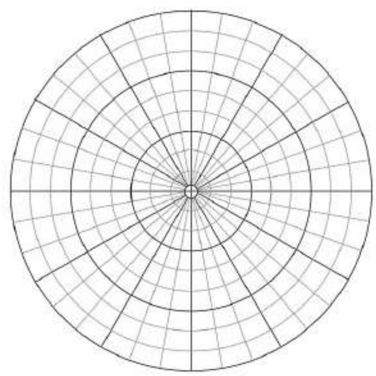


Sketch the indicated graphs.

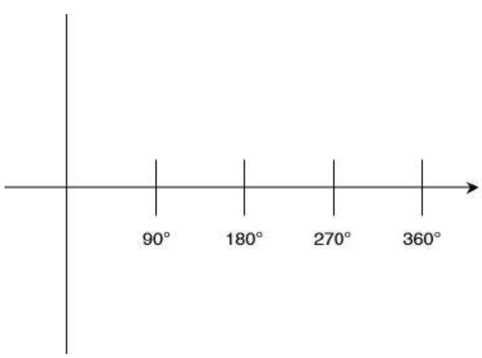
13. $y = 7\sin\theta$



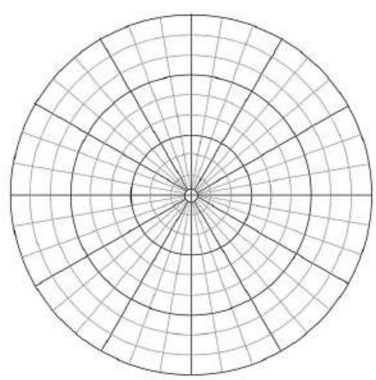
14. $r = 7\sin\theta$



15. $y = 4\sin(3\theta)$



16. $r = 4\sin(3\theta)$



GO

Topic: Using the definition of logarithm to solve for x.

Use the definition of $log\ x$ to find the value of x. Recall that $log\ x$ has a base of 10.
(NO CALCULATORS)

17. $log\ x = 3$

18. $log\ x = -4$

19. $log\ x = 1$

20. $log\ x = 0$

21. $log\ x = \frac{1}{2}$

22. $log\ 10^7 = x$

23. $log\ 0.001 = x$

24. $log\ 1{,}000{,}000 = x$

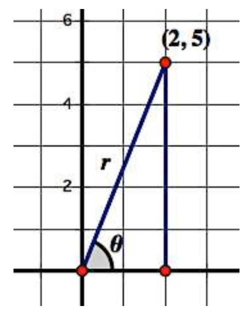
25. $log\ 10^e = x$

7.11H Complex Polar Forms

A Solidify Understanding Task

Alyce and Veronica recall that they have learned how to represent complex numbers as points or vectors on a complex plane by letting the x-axis represent the real component of the complex number, and the y-axis representing the imaginary component. They are wondering if complex numbers can be related to Javier's work with the polar grid. From the internet, they learn that complex numbers can be expressed in polar form, and that much of the arithmetic of complex numbers can be simplified and enhanced when complex numbers are written in polar form. They are using the following problems to introduce Javier to the key ideas.

1. The point (2, 5) can be plotted on a rectangular grid. Find the polar coordinates for the same point. (Express the angle measure to the nearest tenth of a degree.)



2. In the complex plane, the point (2, 5) represents the complex number $2 + 5i$. In general, any point (a, b) represents a corresponding complex number $a + bi$. How is the horizontal distance a and the vertical distance b represented in polar form? That is, how can we use r and θ to describe the complex number $a + bi$?

The arithmetic of complex numbers from a polar perspective

Multiplying complex numbers:

Alyce and Veronica have learned that when complex numbers are written in complex form, $r \cdot \cos\theta + i \cdot r \cdot \sin\theta$, the product of two complex numbers is easy to find.

"You just multiply the *r's* and add the θ's," Veronica says excitedly.

Javier writes out in symbols what Veronica has claimed:

$$\text{If } a_1 + b_1 i = r_1\cos\theta_1 + ir_1\sin\theta_1 \text{ and } a_2 + b_2 i = r_2\cos\theta_2 + ir_2\sin\theta_2,$$

$$\begin{aligned} \text{then } (a_1 + b_1 i)(a_2 + b_2 i) &= (r_1\cos\theta_1 + ir_1\sin\theta_1)(r_2\cos\theta_2 + ir_2\sin\theta_2) \\ &= (r_1 \cdot r_2)\cos(\theta_1 + \theta_2) + i(r_1 \cdot r_2)\sin(\theta_1 + \theta_2) \\ &= a_3 + b_3 i \end{aligned}$$

Javier doesn't understand how Veronica's claim can be true, that is,

$(r_1\cos\theta_1 + ir_1\sin\theta_1)(r_2\cos\theta_2 + ir_2\sin\theta_2) = (r_1 \cdot r_2)\cos(\theta_1 + \theta_2) + i(r_1 \cdot r_2)\sin(\theta_1 + \theta_2)$

so he decides to try out her rule for a specific example.

3. Javier's experiment:

   a. Pick two complex numbers written in the form $a + bi$ and multiply them together algebraically, as you normally would.

   b. Rewrite both of the complex numbers in polar form.

   c. Multiply the polar forms of the two complex numbers together using Veronica's rule.

   d. Convert the product from polar form back to $a + bi$ form.

   e. Did you get the same result using Veronica's rule as you got in part a?

Javier is more convinced, but would like some proof that Veronica's rule will work all of the time, and not just for the few examples he tried. Veronica says, "As I recall, you have to use the sum and difference identities for sine and cosine to prove it." Javier decides to try to prove Veronica's rule.

4. Javier's proof:

   a. Multiply out $(r_1\cos\theta_1 + ir_1\sin\theta_1)(r_2\cos\theta_2 + ir_2\sin\theta_2)$ as the product of two binomials.

   b. Simplify the results using the sum and difference identities for sine and cosine that you wrote in the task *Double Identity*.

   c. Manipulate your final expression until it matches Veronica's claim.

Dividing complex numbers:

Alyce says, "We also learned that to divide complex numbers you divide the *r's* and subtract the θ*'s*." This claim seems even more amazing to Javier. He decides to repeat the work he did with Veronica's claim.

5. Write out Alyce's claim symbolically.

6. Try out an experiment with Alyce's claim by selecting two complex numbers in the form $a + bi$ and divide them algebraically, as you normally would using the conjugate. Then write the complex numbers in polar form and divide them using Alyce's claim. Do you get the same result?

7. If your examples seem to support Alyce's claim, prove it using trig identities.

Powers and roots of complex numbers:

Javier has a new insight of his own as he thinks about Veronica's rule for multiplying complex numbers in polar form. "Since raising something to the n^{th} power is just like repeated multiplication, we can write a rule for finding powers of a complex number written in polar form which will be much easier than multiplying out $(a + bi)^n$. Sweet!"

8. Finish Javier's rule for $(a + bi)^n = (r \cdot \cos\theta + i \cdot r\sin\theta)^{\,n} =$ ____________________________

Javier is wondering what happens to complex numbers as they are raised to higher and higher powers. He starts with the complex number $1 + \sqrt{3}i = 2\cos 60° + 2i\sin 60°$, since $r = 2$ and $\theta = 60°$ are both nice integer numbers.

9. Use Javier's rule to raise $1 + \sqrt{3}i$ to the following powers by using the polar form of this complex number, $2\cos 60° + 2i\sin 60°$, then convert the resulting complex number in polar form back to the form $a + bi$:

a. $\left(1 + \sqrt{3}i\right)^1 =$

b. $\left(1 + \sqrt{3}i\right)^2 =$

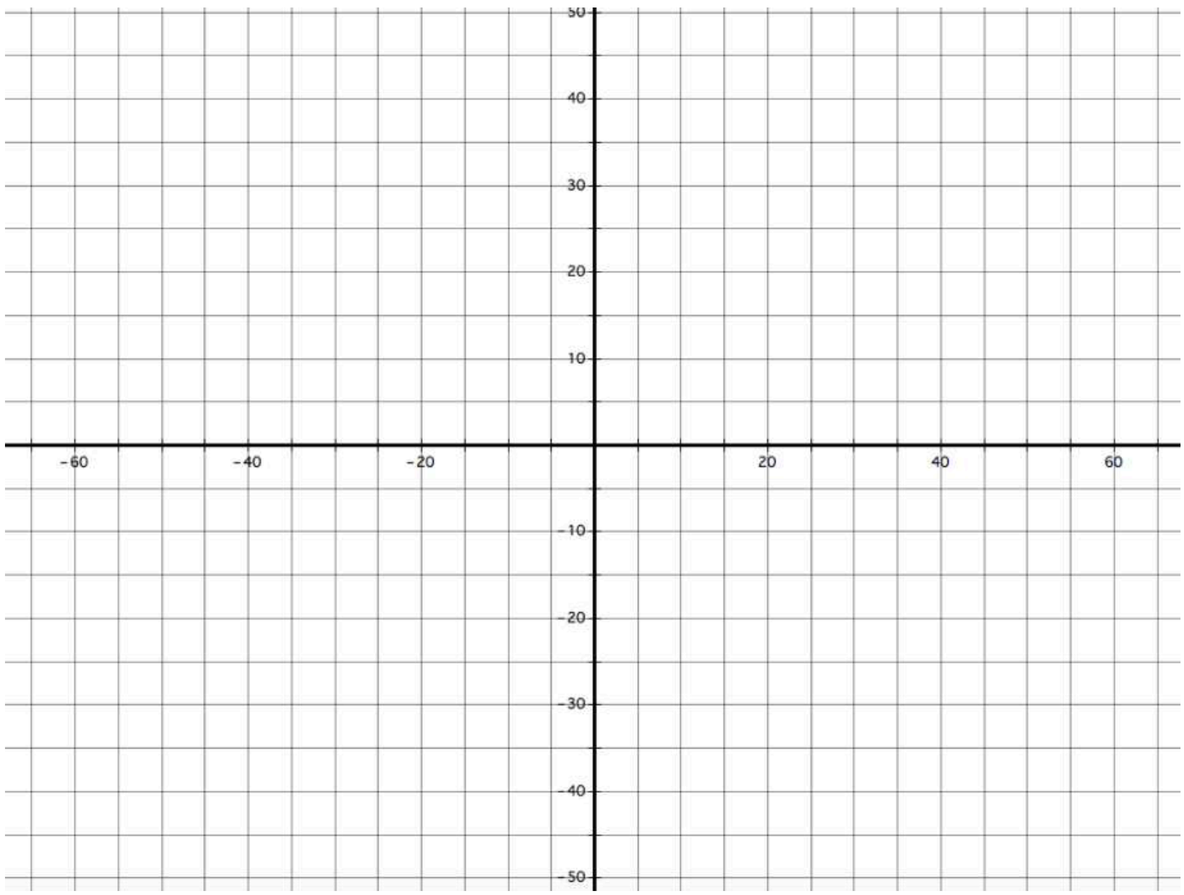
c. $\left(1 + \sqrt{3}i\right)^3 =$

d. $\left(1 + \sqrt{3}i\right)^4 =$

e. $\left(1 + \sqrt{3}i\right)^5 =$

f. $\left(1 + \sqrt{3}i\right)^6 =$

10. Plot each of the above complex numbers on the following complex plane. That is, treat the horizontal axis as a real number axis, and the vertical axis as an imaginary number axis. A complex number $a + bi$ is plotted as the point (a, b). How does your understanding of exponential growth show up in the diagram? In the polar form of the powers?



Javier wonders if his rule can also be used to find roots of complex numbers. The rectangular form of the complex number you found in 9f, $64\cos 360° + 64i\sin 360°$, is 64 + 0$i$. Javier knows that $\sqrt[3]{64} = 4$, since $4 \cdot 4 \cdot 4 = 64$. He decides to try to find the cube root of 64 using his rule applied to the polar form of the complex number 64 + 0i.

11. Find the cube root of 64 using Javier's rule.

$$\sqrt[3]{64 + 0i} = (64\cos 360° + 64i\sin 360°)^{\frac{1}{3}} =$$

Verify that your answer is a cube root of 64 by multiplying it by itself three times:

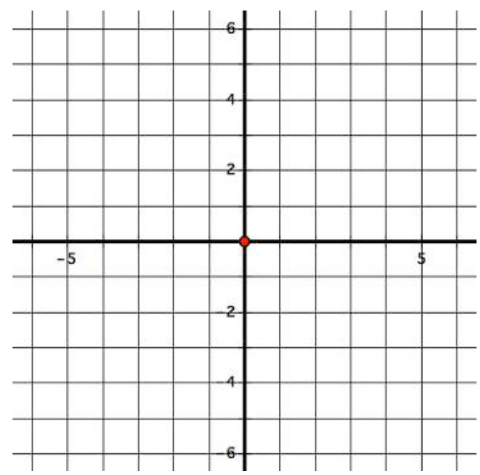
Javier is puzzled by the above results. He knows that $\sqrt[3]{64} = 4$, but this work with complex numbers suggests that $\sqrt[3]{64} = -2 + 2\sqrt{3}i$ is also a cube root of 64 if he considers complex numbers as possible cube roots. He wonders if he is missing any others. Javier decides to do some of his own searching on the internet, to see if he can find an answer to his question. Here are some of the results of his search:

Idea #1: If $x^n = a + bi$, we expect n complex roots of $a + bi$.

Idea #2: The n roots are $\dfrac{360°}{n}$ apart.

Idea #3: $(r\cos\theta + i \cdot r\sin\theta)^{\frac{1}{n}} = \left(r^{\frac{1}{n}} \cdot \cos\left(\frac{k\theta}{n}\right) + i \cdot r^{\frac{1}{n}} \cdot \sin\left(\frac{k\theta}{n}\right)\right) \quad for\ k = 1, 2, 3, \cdots, n$

12. Based on these ideas, what are the three cubes roots of 64?



Plot the cube roots of 64 on the complex plane:

7.11H Complex Polar Forms – Teacher Notes

A Solidify Understanding Task

Purpose: In this task students are introduced to the polar form of complex numbers, $rcos(\theta) + irsin(\theta)$, and they explore procedures for multiplying and adding complex numbers in polar form, taking roots of complex numbers in polar form, and raising complex numbers in polar form to powers. The sum and difference trigonometric identities for sine and cosine are used to verify these procedures.

Core Standards Focus:

N.CN.4. (+) Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

N.CN.5. (+) Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation. For example, $(-1 + \sqrt{3}\,i)^3 = 8$ because $(-1 + \sqrt{3}\,i)$ has modulus 2 and argument 120°.

MVP Honors Standards:

- Define and use polar coordinates and relate them to Cartesian coordinates.
- Represent complex numbers in rectangular and polar form, and convert between rectangular and polar form.
- Translate equations in Cartesian coordinates into polar coordinates and graph them in the polar coordinate plane.
- Multiply complex numbers in polar form and use DeMoivre's Theorem to find roots of complex numbers.

Related Standards: N.CN.1, N.CN.2, N.CN.3

SMP 3 – Construct viable arguments and critique the reasoning of others

Vocabulary: In this task students will be introduced to the *polar form of complex numbers,* $r cos(\theta) + i r sin(\theta)$.

Launch:

Point out to students that in this task we will always consider *r* as having a positive value. That is, *r* is the magnitude of the vector representing the complex number (also known as the modulus). Work through questions 1 and 2 together, and make sure that students understand how to represent a complex number in polar form.

Review Veronica's claim with the students, and point out that the *r's* she is referring to is the *magnitude* of the complex vectors. In the task, Javier writes out Veronica's claim in symbols. Don't let students get hung up with this notation initially. It will be important to revisit this notation in order to prove Veronica's claim. Rather, have students think about Veronica's claim in words, and have them test it out for a problem of their own (see question 3): Pick any two complex numbers in *a* + *bi* form and multiply them together algebraically, using the distributive property; then convert the complex numbers to polar form and apply Veronica's claim that you can multiply the polar forms by *multiplying the magnitudes* and *adding the angles;* verify that the resulting complex number in polar form is equivalent to the complex number found as the product in *a* + *bi* form.

Explore (Small Group):

Give students a few minutes to test out Veronica's claim for their chosen multiplication problem. There should be several different examples produced suggesting that the claim is true. Then have students consider how they might use their trigonometric identities to prove Veronica's claim is always true. Suggest that they return to Javier's notation and identify where there is an unsupported gap in the flow of the algebra from one line of his work to the next. Javier used Veronica's claim, rather than algebra to move from one line to the next, as indicated below. See if students can figure out how to use their trig identities and algebra to fill in this gap so the logic flows from one step to the next.

If $a_1 + b_1 i = r_1 cos\theta_1 + ir_1 sin\theta_1$ *and* $a_2 + b_2 i = r_2 cos\theta_2 + ir_2 sin\theta_2$,

then $(a_1 + b_1 i)(a_2 + b_2 i) = (r_1 cos\theta_1 + ir_1 sin\theta_1)(r_2 cos\theta_2 + ir_2 sin\theta_2)$

⇐ Fill in this gap in logic to form a complete proof. (See question 10 for suggestions.)

$= (r_1 \cdot r_2)cos(\theta_1 + \theta_2) + \mathrm{i}(r_1 \cdot r_2)\sin(\theta_1 + \theta_2)$

$= a_3 + b_3 i$

Once students have completed this proof, they should be able to form a similar chain of reasoning for division of complex numbers based on Alyce's claim (see questions 5-7).

Also allow students to explore the raising to powers and finding roots of complex numbers in polar form. Make note of the progress they are making on these ideas, and select work of students that accurately reflects raising complex numbers to powers as repeated multiplication. Use this work to start the whole class discussion.

Discuss (Whole Class):

Ask, "How does our work with multiplying two complex numbers in polar form relate to the work of raising complex numbers to a power?" Press students to identify that we are just doing repeated multiplication, and based on Veronica's rule we would need to multiply r by itself n times and add θ to itself n times in order to multiply a complex number by itself n times.

Next, examine a student's complex plane diagram of the powers of $\left(1+\sqrt{3}i\right)^n$ for n = {1, 2, 3, 4, 5, 6}, and press students to describe the details of the spiral effect produced, such as how the powers occur at equally spaced angles around the grid, and how the magnitude of the complex powers are growing exponentially (see questions 8-10).

Finally, help students see how they can use their knowledge of the relationship between powers and roots to find different complex roots of 64 (see questions 11 and 12).

Aligned Ready, Set, Go: Trigonometric Functions, Equations and Identities 7.11H

READY, SET, GO! Name Period Date

READY

Topic: Recalling the complex plane

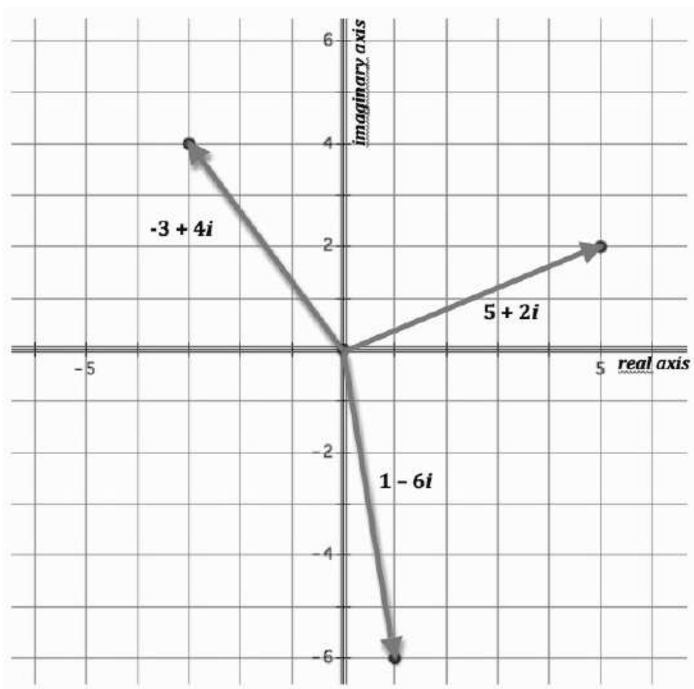
Recall from previous courses, that just as real numbers can be represented by points on the real number line, you can represent a complex number $\boldsymbol{z = a + bi}$ as the ordered pair (a, b) in a coordinate plane called the **complex plane**. The horizontal axis is called the **real axis** and the vertical axis is called the **imaginary axis**. A complex number $a + bi$ can also be represented by a position vector with its tail located at the point (0, 0) and its head located at the point (a, b), as shown in the diagram. It will be useful to be able to move back and forth between both geometric representations of a complex number in the complex plane—sometimes representing the complex number as a single point, and sometimes as a vector.

In the diagram, 3 complex numbers have been graphed as vectors. Rewrite each complex number as a point in the form (a, b).

1. -3 + 4i graphs as (   )
2. 5 + 2i graphs as (   )
3. 1 – 6i graphs as (   )

On the diagram, graph the following complex numbers as vectors.

4. $-5 - 3i$
5. $2 + 4i$
6. $-6 + i$
7. $2 - i$



Topic: Defining the modulus of a complex number

We can compare the relative magnitudes of complex numbers by determining how far they lie away from the origin in the complex plane. We refer to the magnitude of a complex number as its **modulus** and symbolize this length with the notation $|a + bi|$ where $|a + bi| = \sqrt{a^2 + b^2}$.

8. Find the modulus of each of the complex numbers in problems 1 - 7.

1) 2) 3) 4)

5) 6) 7)

SET

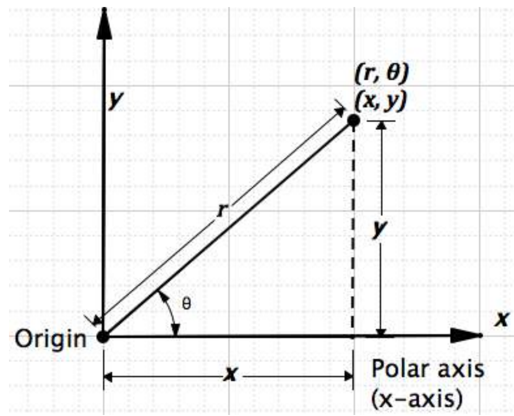
Topic: Connecting polar coordinates and rectangular coordinates

Coordinate Conversion:

The polar coordinates (r, θ) are related to the rectangular coordinates (x, y) as follows:

$\boldsymbol{x = r\cos\theta}$ $\boldsymbol{y = r\,sin\,\theta}$ $\boldsymbol{\tan\theta = \frac{y}{x}}$

$\boldsymbol{r^2 = x^2 + y^2}$



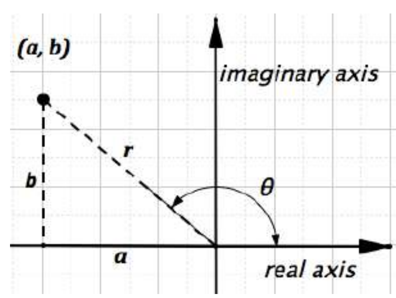
Convert the points from polar to rectangular coordinates.

9. $\left(4, \frac{\pi}{2}\right)$ 10. $\left(\sqrt{3}, \frac{5\pi}{6}\right)$ 11. $(2, \pi)$ 12. $\left(5\sqrt{2}, \frac{7\pi}{4}\right)$

Convert the points from rectangular to polar coordinates.

13. $(-3, 3)$ 14. $(-6, 0)$ 15. $(1, \sqrt{3})$ 16. $(-4\sqrt{3}, -4)$

Topic: Writing the trigonometric form of a complex number



Consider the complex number $a + bi$. The angle θ is the angle measure from the positive real axis to the line segment connecting the origin and the point (a, b).

$\boldsymbol{a = r\cos\theta \;\; and \;\; b = r\sin\theta, where\; r = \sqrt{a^2 + b^2}.}$

By replacing a and b, you have $\boldsymbol{a + bi = (r\cos\theta) + (r\sin\theta)i}$. Factor out the r to obtain the **trigonometric form of a complex number.**

If $\boldsymbol{z = a + bi}$ then the trigonometric form is $\boldsymbol{z = r(\cos\theta + i\sin\theta)}$.

Write the complex numbers in trigonometric form $\boldsymbol{z = r(\cos\theta + i\sin\theta)}$.

17. $-3 - i$
18. $3 - 3i$
19. $-7 + 4i$
20. $\sqrt{3} + i$

Write the complex number in standard form $\boldsymbol{a + bi}$.

21. $3(\cos 120° + \sin 120°)$
22. $5(\cos 135° + i\sin 135°)$
23. $\sqrt{8}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$
24. $8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

GO

Topic: Practicing operations on complex numbers

Perform the indicated operation. Write your answers in standard form.

25. $(4 + 7i) + (12 - 2i)$
26. $(11 - 8i) + (-4 - 3i)$
27. $(10 + 6i) - (16 - 3i)$
28. $(-7 - i) - (9 + i)$
29. $(1 + i)(4 - 2i)$
30. $(5 + 6i)(5 - 6i)$