mathematics vision project

*Transforming Mathematics Education*

# SECONDARY MATH THREE

*An Integrated Approach*

MODULE 8 HONORS

# Modeling With Functions

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**The Mathematics Vision Project**

*Scott Hendrickson, Joleigh Honey, Barbara Kuehl, Travis Lemon, Janet Sutorius*

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# 8.1 Function Family Reunion

## *A Solidify Understanding Task*



During the past few years of math classes you have studied a variety of functions: linear, exponential, quadratic, polynomial, rational, radical, absolute value, logarithmic and trigonometric.

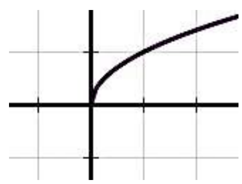
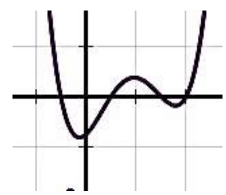
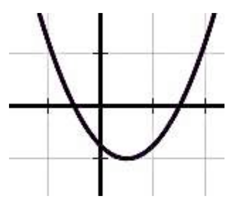
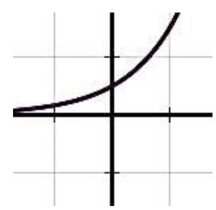
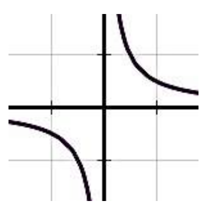
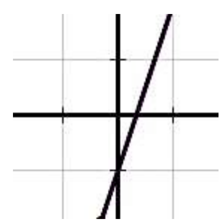
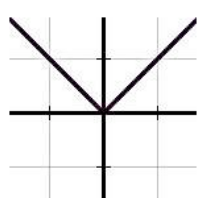
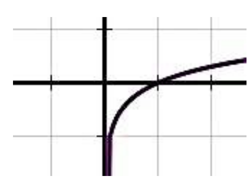
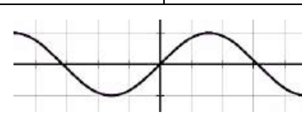
Like a family, each of these types of functions have similar characteristics that differ from other types of functions, making them uniquely qualified to model specific types of real world situations. Because of this, sometimes we refer to each type of function as a "family of functions."

1. Match each function family with the algebraic notation that best defines it.

| ***Function Family Name*** | ***Algebraic Description of the Parent Function*** |
|---|---|
| **1. *linear*** | **A.** $y = \|x\|$ |
| **2. *exponential*** | **B.** $y = a\sin(bx)$ *or* $y = a\cos(bx)$ |
| **3. *quadratic*** | **C.** $y = mx + b$ |
| **4. *polynomial*** | **D.** $y = \log_b(x)$ |
| **5. *rational*** | **E.** $y = ax^2 + bx + c$ |
| **6. *absolute value*** | **F.** $y = \frac{1}{x}$ |
| **7. *logarithmic*** | **G.** $y = ab^x$ |
| **8. *trigonometric*** | **H.** $y = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$ |
| **9. *radical*** | **I.** $y = \sqrt[n]{x}$ |

Just like your family, each member of a function family resembles other members of the family, but each has unique differences, such as being "wider" or "skinnier", "taller" or "shorter", or other features that allow us to tell them apart. We might say that each family of functions has a particular "genetic code" that gives its graph its characteristic shape. We might refer to the simplest form of a particular family as "the parent function" and consider all transformations of this parent function to be members of the same family.

2. Match each function family with the characteristic shape of the graph that fits it.

<table>
<tr><th>Function Family Name</th><th colspan="2">Characteristic Shape of the Graph</th></tr>
<tr><td>1. linear</td><td rowspan="2">A. </td><td rowspan="2">B. </td></tr>
<tr><td>2. exponential</td></tr>
<tr><td>3. quadratic</td><td rowspan="2">C. </td><td rowspan="2">D. </td></tr>
<tr><td>4. polynomial</td></tr>
<tr><td>5. rational</td><td rowspan="2">E. </td><td rowspan="2">F. </td></tr>
<tr><td>6. absolute value</td></tr>
<tr><td>7. logarithmic</td><td rowspan="2">G. </td><td rowspan="2">H. </td></tr>
<tr><td>8. trigonometric</td></tr>
<tr><td>9. radical</td><td colspan="2">I. </td></tr>
</table>

Function family characteristics are passed on to their "children" through a variety of transformations. While the members of each family shares common characteristics, transformations make each member of a family uniquely qualified to accomplish the mathematical work they are required to do.

For each of the following tables, a set of coordinate points that captures the characteristics of a parent graph is given. The additional columns give coordinate points for additional members of the family after a particular transformation has occurred. Write the rule for each of the different transformations of the parent graph. (Note: We can think of each new set of coordinate points (that is, the *image* points) as a geometric transformation of the original set of coordinate points (that is, the *pre-image* points) and use the notation associated with geometric transformations to describe transformation. Or, we can write the rule using algebraic function notation. Use both types of notation to represent each transformation.)

3.

| | *pre-image* (parent graph) | *image 1* | *image 2* | *image 3* |
|---|---|---|---|---|
| *geometric notation* | $(x, y)$ | $(x, y) \rightarrow (x,\ y + 2)$ | | |
| *function notation* | $f(x) = x^2$ | $f_1(x) = x^2 + 2$ | | |
| selected points that fit this image | (-2, 4) | (-2, 6) | (-2, 8) | (-3, 4) |
| | (-1, 1) | (-1, 3) | (-1, 2) | (-2, 1) |
| | (0, 0) | (0, 2) | (0, 0) | (-1, 0) |
| | (1, 1) | (1, 3) | (1, 2) | (0, 1) |
| | (2, 4) | (2, 6) | (2, 8) | (1, 4) |

4.

| | *pre-image* (parent graph) | *image 1* | *image 2* | *image 3* |
|---|---|---|---|---|
| *geometric notation* | $(x, y)$ | | | |
| *function notation* | $f(x) = 2^x$ | | | |
| selected points that fit this image | (-2, $\frac{1}{4}$) | (-2, 1) | (-2, $-\frac{1}{4}$) | (-3, $\frac{1}{4}$) |
| | (-1, $\frac{1}{2}$) | (-1, 2) | (-1, $-\frac{1}{2}$) | (-2, $\frac{1}{2}$) |
| | (0, 1) | (0, 4) | (0, -1) | (-1, 1) |
| | (1, 2) | (1, 8) | (1, -2) | (0, 2) |
| | (2, 4) | (2, 16) | (2, -4) | (1, 4) |

5.

| | *pre-image* (parent graph) | *image 1* | *image 2* | *image 3* |
|---|---|---|---|---|
| *geometric notation* | $(x, y)$ | | | |
| *function notation* | $f(x) = \|x\|$ | | | |
| selected points that fit this image | (-2, 2) | (-2, -4) | (2, 2) | (-5, 2) |
| | (-1, 1) | (-1, -2) | (3, 1) | (-4, 1) |
| | (0, 0) | (0, 0) | (4, 0) | (-3, 0) |
| | (1, 1) | (1, -2) | (5, 1) | (-2, 1) |
| | (2, 2) | (2, -4) | (6, 2) | (-1, 2) |

6.

| | *pre-image* (parent graph) | *image 1* | *image 2* | *image 3* |
|---|---|---|---|---|
| *geometric notation* | $(x, y)$ | | | |
| *function notation* | $f(x) = \sin(x)$ | | | |
| selected points that fit this image | (0, 0) | (0, 2) | (0, 0) | (0, 0) |
| | ($\pi/2$, 1) | ($\pi/2$, 3) | ($\pi/4$, 1) | ($\pi/2$, -2) |
| | ($\pi$, 0) | ($\pi$, 2) | ($\pi/2$, 0) | ($\pi$, 0) |
| | ($3\pi/2$, -1) | ($3\pi/2$, 1) | ($3\pi/4$, -1) | ($3\pi/2$, 2) |
| | ($2\pi$, 0) | ($2\pi$, 2) | ($\pi$, 0) | ($2\pi$, 0) |

7.

| | *pre-image* (parent graph) | *image 1* | *image 2* | *image 3* |
|---|---|---|---|---|
| *geometric notation* | $(x, y)$ | | | |
| *function notation* | $f(x) = \sqrt{x}$ | | | |
| selected points that fit this image | (0, 0) | (0, 0) | (0, 0) | (3, 0) |
| | (1, 1) | (1, ½) | (½, 1) | (4, 1) |
| | (4, 2) | (4, 1) | (2, 2) | (7, 2) |
| | (9, 3) | (9, $3/2$) | ($9/2$, 3) | (12, 3) |
| | (16, 4) | (16, 2) | (8, 4) | (19, 4) |

# 8.1 Function Family Reunion – Teacher Notes

## *A Solidify Understanding Task*

**Purpose**: Students have previously studied a variety of functions: linear, exponential, quadratic, polynomial, rational, radical, absolute value, logarithmic and trigonometric. They have made many observations about how changing the values of the parameters in a function transform the graph. For example, quadratic functions of the form $y = ax^2 + bx + c$ shift up or down as the value of $c$ changes. Likewise, trigonometric functions of the form $y = a\sin(bx) + c$ also shift up or down as the value of $c$ changes. Students have also observed other transformations of graphs, such as shifting left or right, stretching or shrinking vertically or horizontally, or reflecting over the $x$-axis, for each type of function. They have noticed that similar transformations take place across various types of functions for similar parameter changes. The purpose of this task is to solidify observations about transformations that have surfaced in previous modules, to begin to connect geometric transformations to function transformations, and to confirm that transformations are consistent across all types of functions. This review of function transformations sets the stage for combining various types of functions by addition, subtraction, multiplication, division and composition in order to model more complex relationships between quantities, which will be the focus of the remainder of this module.

**Core Standards Focus**:

**F.BF.3** Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

**G.CO.2** Describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

**Standards for Mathematical Practice:**

**SMP 7 – Look for and make use of structure**

**Vocabulary:** This task refers to *function families* and *parent functions* as a way of relating functions that share common characteristics when one function is a transformation of the other.

**The Teaching Cycle:**

**Launch (Whole Class):**

Introduce the image of a "family" of functions as a way of classifying functions that share common characteristics, "parent functions" as the simplest member of the family, "children" as transformations of the parent function, and a set of key points that define the shape of the parent function as the "genetic code." Give students a few minutes to complete the matching activities in questions 1 and 2 to reactivate students' background knowledge about the various function types they have studied. Point out that in question 1 the most important parameters for the function type have been included in the notation of the algebraic description of the function family, rather than just giving the simplest example, but that other parameters might still be used even with this more detailed algebraic description. For example, it is more useful to describe linear functions with the notation $y = mx + b$, instead of just giving the simplest member of this family, $y = x$. We would consider the "parent" function for linear functions to be $y = x$, but a more useful description of linear functions includes the parameters $m$ and $b$ in order to capture important information about the graph (slope and $y$-intercept) or about the context it might model (rate of change and start amount).

Once you have reviewed the various function types, their algebraic descriptions and their characteristic graphs, turn students' attention to question 3. Remind students that when we studied geometric transformations we used notation like $(x, y) \to (x + 3,\ y + 2)$ or $(x, y) \to (2x, 2y)$, to describe transformations such as translations and dilations. On the other hand, we have used function notation to describe moving and altering function graphs on a coordinate grid. Since a function is a collection of points, we can consider a function transformation from either a geometric or algebraic perspective.

As a class, examine the set of points labeled *pre-image* and *image 1* in the first table for the function, $f(x) = x^2$, and verify that students understand the notation being used. Point out the use of the subscript on the name of the function, $f_1(x)$, to distinguish it from the parent function $f(x)$.

Assign students to fill out the geometric and function notation that correctly describes each transformed image in each table. Remind students that the first column of data points is the *pre-image* points for all three transformations described in the columns labeled *image 1, image 2* and *image 3*.

**Explore (Small Group):**
As students work on the tables in question 3, make sure they are treating the first column of data points as the *pre-image* points for all three transformations described in the columns labeled *image 1, image 2* and *image 3* for each table. You may need to ask students, "What has changed between the data points in the pre-image and the image? Have the *x*-coordinates changed, or the *y*-coordinates?" In this set of problems, only one of the *x* or *y*-coordinates have been changed in order to focus attention on the differences between keeping the *x*-coordinates constant and changing *y* (creating a vertical translation or dilation) or keeping the *y*-coordinates constant and changing *x* (creating a horizontal translation or dilation).

Listen for how students notice the connections between the geometric and function notation when either the *x* or *y* coordinates change. Listen for how students resolve (or continue wondering about) the function notation for the horizontal transformations. Identify students who can make an argument for why we subtract a number from *x* in the geometric notation when we want to translate the points to the left, but we add the same number to *x* in the function notation when we want to shift the graph to the left. Also identify students who can make an argument for why we multiple *x* by a number in the geometric notation when we want to stretch or shrink the horizontal distance between points, but we divide *x* by that same number in the function notation to create the same effect. Prompt students to think about the ordered-pairs as giving inputs and outputs of a function, and ask what would need to happen to the *x*-values so the outputs remain constant.

**Discuss (Whole Class):**

You do not need to present all of the tables, unless you have time to do so. The absolute value function table would be a particularly good one to discuss, since it includes two examples of a horizontal translation (image 2 and image 3) and can be easily understood in terms of the question, "If I want to get this output, what do I need to do with the input?" For example, to get 0 out of the absolute value function when we put 4 in, we would need to subtract 4 from the input before taking the absolute value. Therefore, $f_2(x) = |x - 4|$ is the function notation for the transformation that can be described geometrically as $(x, y) \rightarrow (x + 4, y)$. The radical function table would also be a good one to discuss, since it contains both a horizontal translation (image 3) and a horizontal dilation (image 2).

If there is additional time, assign different groups to present their notation for each image in the remaining tables. Resolve any issues or misconceptions that arise, by allowing students who disagree to share their reasoning about the tables.

**Aligned Ready, Set, Go: Modeling with Functions 8.1**

READY, SET, GO! Name Period Date

## READY

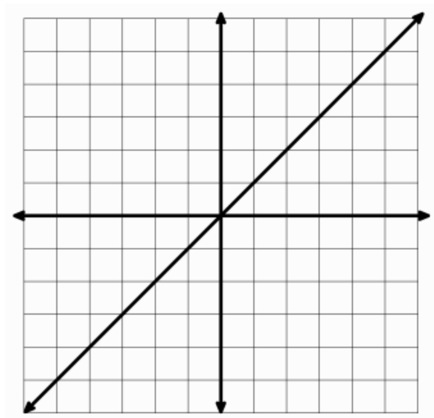
Topic: Reviewing transformations

1. Graph the following linear equations on the grid. The equation y = x has been graphed for you. For each new equation explain what the number 2 does to the graph of y = x. Pay attention to the y-intercept, the x-intercept, and the slope. Identify what changes in the graph and what stays the same.

   a. $y_1 = x + 2$

   b. $y_2 = x - 2$

   c. $y_3 = 2x$



2. Graph the following quadratic equations on the grid. The equation $y = x^2$ has been graphed for you. For each new equation explain what the number 3 does to the graph of $y = x^2$. Pay attention to the y-intercept, the x-intercept(s), and the rate of change. Identify what changes in the graph and what stays the same.

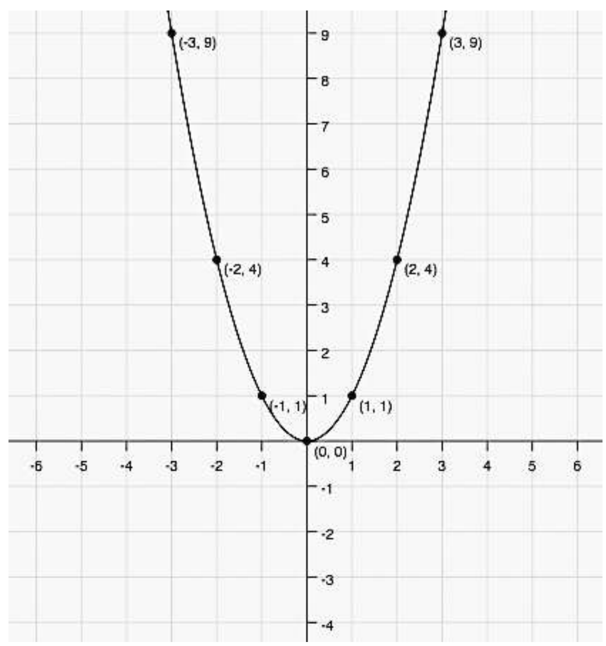
   a. $y_1 = x^2 + 3$

   b. $y_2 = x^2 - 3$

   c. $y_3 = (x - 3)^2$

   d. $y_4 = (x + 3)^2$

   e. $y_5 = 3x^2$

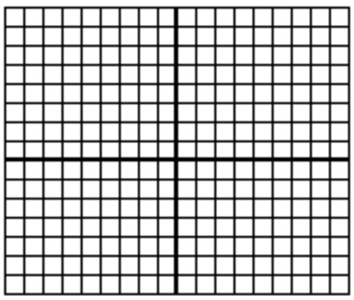


## SET

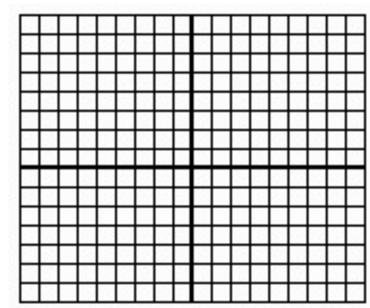
Topic: Transforming parent functions

**Sketch the graph of the parent function and the graph of the transformed function on the same set of axes.**

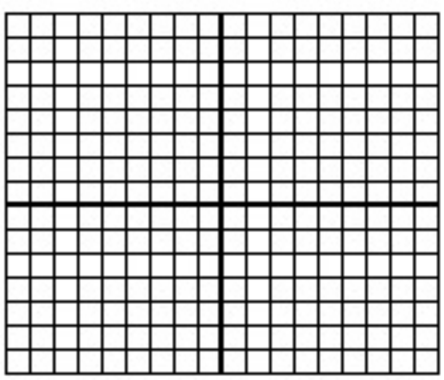
3. $f(x) = |x|, and\ g(x) = |x+3|$



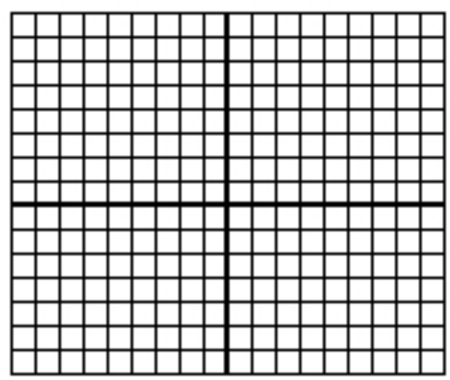
4. $h(x) = 2^x, and\ j(x) = 2^{-x}$



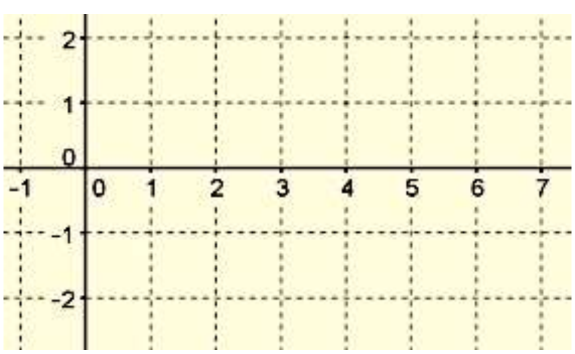
5. $r(x) = x^2, and\ s(x) = -\frac{1}{2}x^2 + 5$



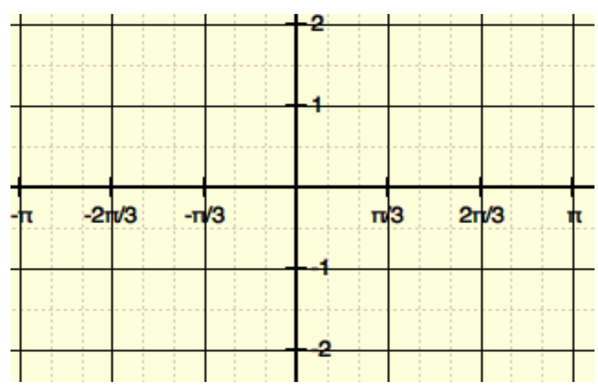
6. $v(x) = \frac{1}{x}, and\ w(x) = -\frac{1}{x}$



7. $k(x) = log(x), and\ m(x) = -1 + log(x)$




8. $p(x) = sin(x), and\ q(x) = 2sin\left(x + \frac{\pi}{3}\right)$



## GO

Topic: Evaluating functions

**Find the function values: $f(-2), f(0), f(1), and\ f(3)$. Indicate if the function is undefined for a given value of x.**

9. $f(x) = |x+5|$

10. $f(x) = |x-2|$

11. $f(x) = x|x|$

12. $f(x) = 3^x$

13. $f(x) = 3^{x+2}$

14. $f(x) = (3^x) + x$

15. $f(x) = \frac{x}{x}$

16. $f(x) = \frac{x}{(x-4)}$

17. $f(x) = \frac{x}{(x+2)} - 5$

18. $f(x) = log_3 x$

19. $f(x) = log_7 (7)^x$

20. $f(x) = x log_{10} 1000$

# 8.2 Imagineering

## *A Develop Understanding Task*



You are excited to get to vote on the plans for a proposed new thrill ride at a local theme park. The engineers want public input on the design for the new ride. You are one of ten teenagers who have been selected to review the plans based on your good math grades!

As your excitement mounts, the engineers begin their presentation. To your dismay, there are no models or illustrations of the proposed rides—each ride is described only with equations. The equations represent the path a rider would follow through the course of the ride.

Unfortunately, your cell phone—which contains a graphing calculator app—is completely discharged due to too much texting and surfing the internet. So, you are trying hard to keep up with the presentation by trying to imagine what the graphs of each of these equations would look like. While each equation consists of functions you are familiar with, the combination of functions in each equation has you wondering about their combined effects.

For each of the following proposed thrill rides, use your imagination and best reasoning about the individual functions involved to sketch a graph of the path of the rider. Let $y$ represent the height of the rider above the ground and $x$ represent the distance from the start of the ride. Explain your reasoning about the shape of the graph. (Note: Use radians for trigonometric functions.)

**Proposal #1: "*The Mountain Climb*"**

The Equation: $y = 2x + 5\sin(x)$

My Graph:

My Explanation:

**Proposal #2: "*The Periodic Bump*"**

The Equation: $y = |10\sin(x)|$

My Graph:

My Explanation:

**Proposal #3: "*The Amplifier*"**

The Equation: $y = x \cdot \sin(x)$

My Graph:

My Explanation:

**Proposal #4: "*The Gentle Wave*"**

The Equation: $y = 10(0.9)^x \cdot \sin(x)$

My Graph:

My Explanation:

**Proposal #5: "*The Spinning High Dive*"**

The Equation: $y = 100 - x^2 + 5\sin(4x)$

My Graph:

My Explanation:

When you got home your friends were all anxiously waiting to hear about the proposed new rides. After explaining the situation, your friends all pull out their calculators and they began comparing your imagined images with the actual graphs.

Some of your friends' graphs differed from the others because of their window settings. Some window settings revealed the features of the graphs you were expecting to see, while other window settings obscured those features.

Examine the actual graphs of each of the thrill ride proposals. Select a window setting that will reveal as many of the features of the graphs as possible. Explain any differences between your imagined graphs and the actual graphs. What features did you get right? What features did you miss?

**Proposal #1: "*The Mountain Climb*"**

The Equation: $y = 2x + 5\sin(x)$

Actual Graph:

What features I got right and what I missed:

**Proposal #2: "*The Periodic Bump*"**

The Equation: $y = |10\sin(x)|$

Actual Graph:

What features I got right and what I missed:

**Proposal #3: "*The Amplifier*"**

The Equation: $y = x \cdot \sin(x)$

Actual Graph:

What features I got right and what I missed:

**Proposal #4: "*The Gentle Wave*"**

The Equation: $y = 10(0.9)^x \cdot \sin(x)$

Actual Graph:

What features I got right and what I missed:

**Proposal #5: "*The Spinning High Dive*"**

The Equation: $y = 100 - x^2 + 5\sin(4x)$

Actual Graph:

What features I got right and what I missed:

You and your friends decide to propose a different ride to the engineers. Name your proposal and write its equation. Explain why you think the features of this graph would make a fun ride.

**My Proposed Ride:**

The equation for my ride:

My explanation of my proposal:

# 8.2 Imagineering – Teacher Notes

## *A Develop Understanding Task*

**Purpose**: In this task students surface ways of thinking about the effects of combining standard function types using arithmetic operations such as adding functions or multiplying functions together. Students anticipate how the features of each function will show up in the graph of the combined functions by first predicting the shape of the graphs, and then comparing their predictions to the actual graphs. Students will work to resolve discrepancies between their predictions and the actual graphs, and in so doing will consider strategies for analyzing arithmetic combinations of functions.

**Core Standards Focus**:

**F.BF.1** Write a function that describes a relationship between two quantities.★

**F.BF.1b** Combine standard function types using arithmetic operations. For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.

**Standards for Mathematical Practice:**

**SMP 2 – Reason abstractly and quantitatively**

**Vocabulary:** This task introduces students to *arithmetic combinations* of functions and *function composition*.

**The Teaching Cycle:**

**Launch (Whole Class):**

Begin by letting students describe some of their favorite amusement park rides, and then introduce them to the context of this task. Inform students that they should use "reasoned imagination" to sketch each of the graphs based on their knowledge of the individual functions involved and the meaning of the operations used to combine the functions. They should not worry so much about accurately plotting points, but rather they should focus on trying to capture the features of the individual functions and how they imagine they would work together. They should focus on shape and form rather than perfecting the details. Assign students to work individually for a few minutes making their own predictions for the shape of each graph and writing their explanations.

**Explore (Small Group):**
Make sure that students are using their "reasoned imagination" and not calculators on the first part of the task. It may help to assure them that they will eventually be able to compare their graphs to the actual graphs in the second part of the task. Watch for the reasoning they give for the shapes of the graphs and prompt for that reasoning by asking questions such as, "How is the effect of the exponential function showing up in your graph?" If students are stuck, ask a question such as, "If this term (or factor) was a number instead of a function, what effect would it have on the graph?" It may help some students to switch the terms in an expression. For example, when the expression $2x + 5\sin(x)$ is rewritten as $5\sin(x) + 2x$ it may remind them of the form $a\sin(bx) + c$, where $c$ gives the location of the midline. They may then realize that the "middle" of the sine curve is constantly increasing, and sketch a graph to represent this new characteristic.

Watch for misconceptions in student work, such as graphs that aren't functions (e.g., the *Spinning High Dive* may be conceived as a curve spiraling along a parabolic path) or the absolute value of a function going below the $x$-axis.

After students have made their own predictions, allow them to compare their graphs with other members of their group. Launch the second part of the task by discussing how different choices of window settings may reveal or obscure the features of the graphs, and suggest that they experiment with different windows before recording the actual graphs. Let students work together as they sort out what might be an appropriate window. Ask questions to get students to probe more deeply into the shapes of the graphs, such as, "How is the amplitude of the graph affected when the parameter $a$ is a function instead of a number?"

**Discuss (Whole Class):**

Select students to present their initial predictions and explanations, and how those predictions were confirmed or refuted when they examined the actual graphs. Allow other students to add to the descriptions of each graph. The discussion should surface strategies for analyzing combinations of functions by considering how replacing a parameter in a standard function (such as the *a, b, c* or *m* in each of the following function types: $y = a\sin(bx) + c$, $y = ab^x$, $y = mx + b$ or $y = ax^2$) with another function will affect the same feature of the graph as represented by the parameter which was replaced.

Window settings can obscure some of the features of the graphs you want students to notice. Here are some suggested window settings for each of the given equations.

Equation #1: $0 \leq x \leq 20,\ 0 \leq y \leq 40$

Equation #2: $0 \leq x \leq 20, -5 \leq y \leq 15$

Equation #3: $0 \leq x \leq 20, -30 \leq y \leq 30$

Equation #4: $0 \leq x \leq 20, -10 \leq y \leq 10$

Equation #5: $0 \leq x \leq 10,\ 0 \leq y \leq 120$

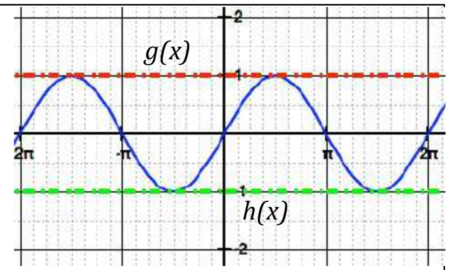
**Aligned Ready, Set, Go: Modeling with Functions 8.2**

READY, SET, GO! Name Period Date

## READY

Topic: Exploring the boundaries of functions

1. The curve in the graph at the right shows the graph of $f(x) = sin(x)$. (blue)

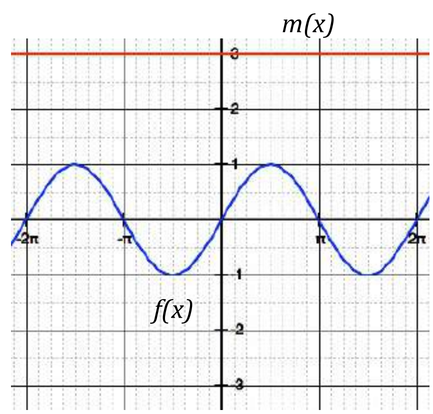



   a. Write the equation of the dotted line labeled $g(x)$. (red)

   b. Write the equation of the dotted line labeled $h(x)$. (green)

   c. List everything you notice about the graphs of $f(x), g(x), and\ h(x)$.

2. The curve in the graph at the right shows the graph of $f(x) = sin(x)$.



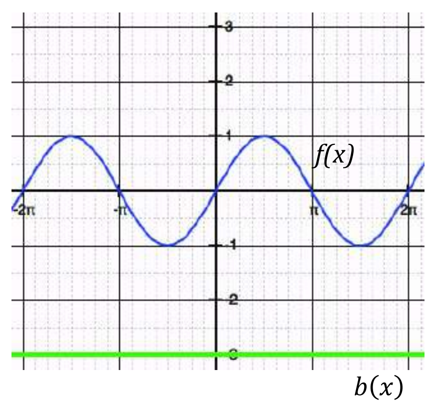

   a. Write the equation of the line labeled $m(x)$. (red)

   b. Sketch in the graph of $f(x) * m(x)$.

   c. What is the equation of $f(x) * m(x)$?

   d. Would the line y = -3 also be a boundary line for your sketch? Explain.

3. The curve in the graph at the right shows the graph of $f(x) = sin(x)$. (blue)




   a. Write the equation of the line labeled $b(x)$. (green)

   b. Sketch in the graph of $f(x) * b(x)$.

   c. What is the equation of $f(x) * b(x)$?

   d. How is the graph of $f(x) * b(x)$ <u>different</u> from the graph of $f(x) * m(x)$?

   e. Would the line y = 3 also be a boundary line for your sketch? Explain.

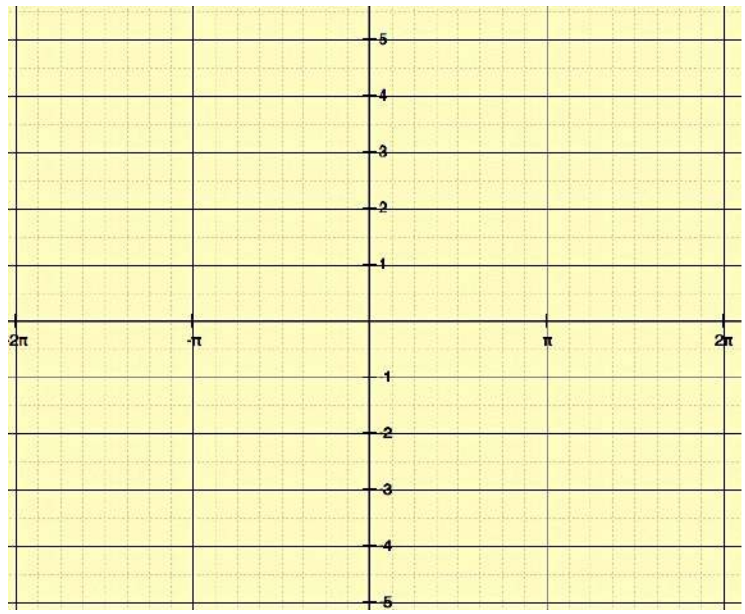
## SET

Topic: Combining functions

4. $f(x) = x$ $\quad g(x) = \sin(x)$ $\quad h(x) = f(x) + g(x)$

Fill in the values for *h(x)* in the table. Then graph $h(x) = x + sin(x)$ with a smooth curve.

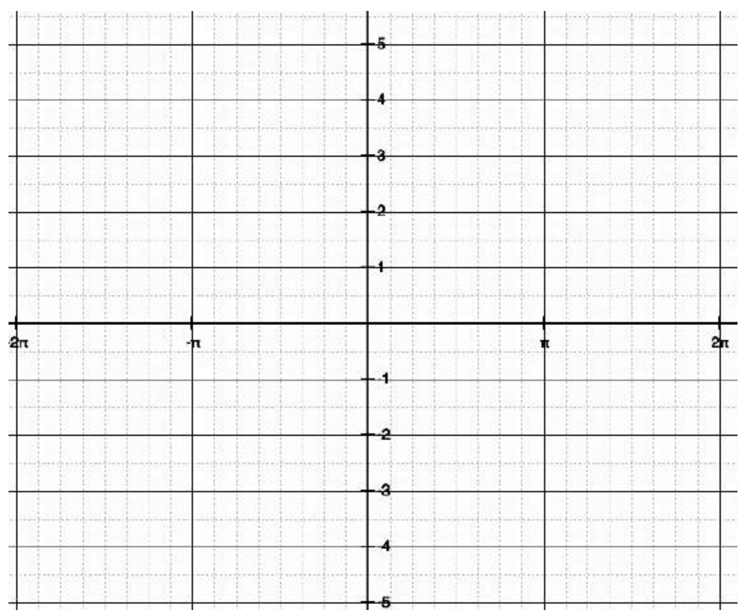
| x | *f (x)* | *g(x)* | *h(x)* |
|---|---|---|---|
| -2π | -6.28 | 0 | |
| $-\frac{3\pi}{2}$ | -4.71 | 1 | |
| -π | -3.14 | 0 | |
| $-\frac{\pi}{2}$ | -1.57 | -1 | |
| 0 | 0 | 0 | |
| $\frac{\pi}{2}$ | 1.57 | 1 | |
| π | 3.14 | 0 | |
| $\frac{3\pi}{2}$ | 4.71 | -1 | |
| 2π | 6.28 | 0 | |




5. $f(x) = x$ $\quad g(x) = \sin(x)$

Now graph $k(x) = f(x) * g(x)$ or $k(x) = x * sin(x)$

| x | *f (x)* | *g(x)* | *k(x)* |
|---|---|---|---|
| -2π | -6.28 | 0 | |
| $-\frac{3\pi}{2}$ | -4.71 | 1 | |
| -π | -3.14 | 0 | |
| $-\frac{\pi}{2}$ | -1.57 | -1 | |
| 0 | 0 | 0 | |
| $\frac{\pi}{2}$ | 1.57 | 1 | |
| π | 3.14 | 0 | |
| $\frac{3\pi}{2}$ | 4.71 | -1 | |
| 2π | 6.28 | 0 | |



**Match each equation below with the appropriate graph. Describe the features of the graph that helped you match the equations.**

6. $f(x) = |x^2 - 4|$
*key features:*

7. $g(x) = -x + 5\sin(x)$
*key features:*

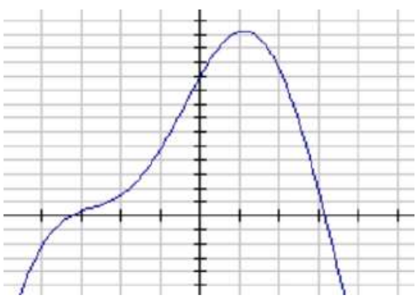
8. $h(x) = 4|sinx|$
*key features:*

9. $d(x) = 10 - x^2 + 5sin(x)$
*key features:*

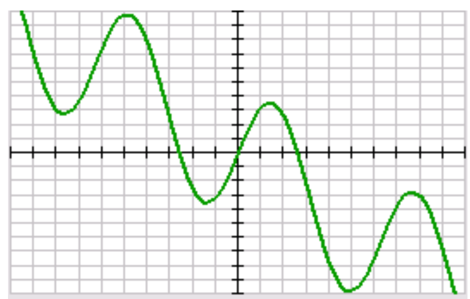
10. $w(x) = -x * 2sin(x)$
*key features:*

11. $r(x) = (2x - 4) + |x|$
*key features:*

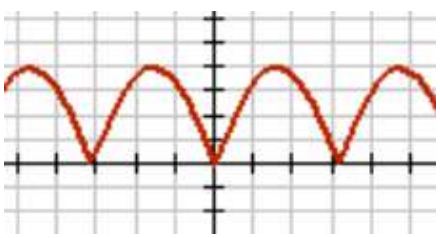
a)



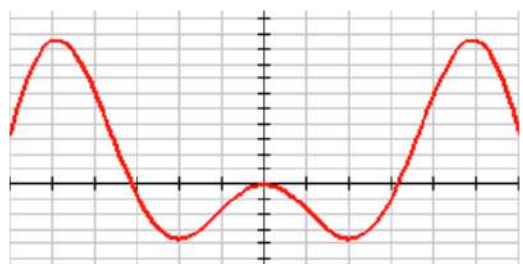
b)



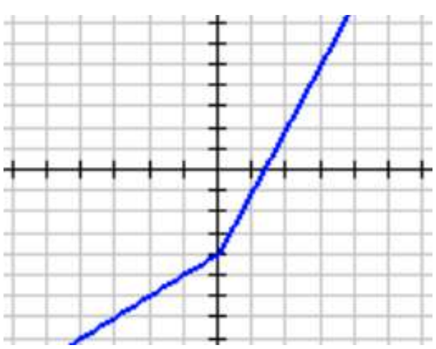
c)



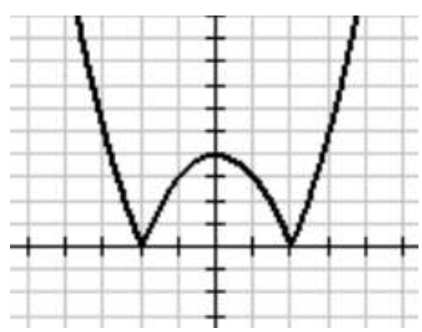
d)



e)



f)



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## GO

Topic: Identifying key features of function families

**The chart below names five families of functions and the parent function. The parent is the equation in its simplest form. In the right hand column is a list of key features of the functions in random order. Match each key feature with the correct function. A key feature may relate to more than one function.**

<table>
<tr><th>Family</th><th>Parent(s)</th><th>Key features</th></tr>
<tr><td>12. Linear</td><td>$y = x$</td><td rowspan="5">a) The ends of the graph have the same behavior.<br>b) The graphs have a horizontal asymptote and a vertical asymptote.<br>c) The graph only has a horizontal asymptote.<br>d) These functions either have both a local maximum and minimum or neither a local maximum and minimum.<br>e) The graph is usually defined in terms of its slope and y-intercept.<br>f) The graph has either a maximum or a minimum but not both.<br>g) As x approaches -∞, the function values approach the x-axis.<br>h) The ends of the graph have opposite behavior.<br>i) The rate of change of this graph is constant.<br>j) The rate of change of this graph is constantly changing.<br>k) This graph has a linear rate of change.<br>l) These functions are of degree 3.<br>m) The variable is an exponent.<br>n) These functions contain fractions with a polynomial in both the numerator and denominator.<br>p) The constant will always be the y-intercept.</td></tr>
<tr><td>13. Quadratic</td><td>$y = x^2$</td></tr>
<tr><td>14. Cubic</td><td>$y = x^3$</td></tr>
<tr><td>15. Exponential</td><td>$y = 2^x$<br>$y = 3^x$<br>etc.</td></tr>
<tr><td>16. Rational</td><td>$y = \frac{1}{x}$</td></tr>
</table>

# 8.3 The Bungee Jump Simulator

### *A Solidify Understanding Task*

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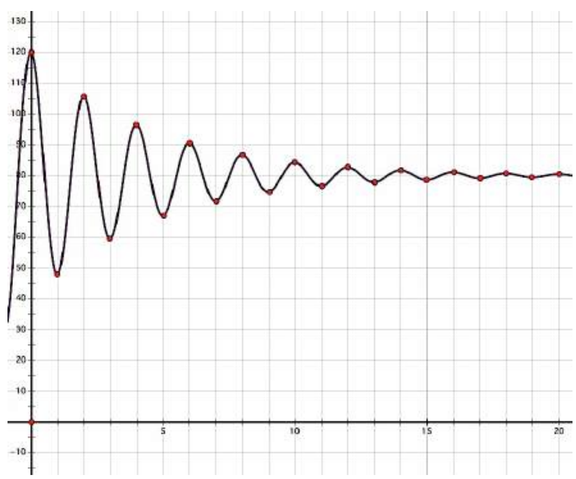
As a reward for helping the engineers at the local amusement park select a design for their next ride, you and your friends get to visit the amusement park for free with one of the engineers as a tour guide. This time you remember to bring your calculator along, in case the engineers start to speak in "math equations" again.

Sure enough, just as you are about to get in line for the *Bungee Jump Simulator*, your guide pulls out a graph and begins to explain the mathematics of the ride. To prevent injury, the ride has been designed so that a bungee jumper follows the path given in this graph. Jumpers are launched from the top of the tower at the left, and dismount in the center of the tower at the right after their up and down motion has stopped. The cable to which their bungee cord is attached moves the rider safely away from the left tower and allows for an easy exit at the right.

Your tour guide won't let you and your friends get in line for the ride until you have reproduced this graph on your calculator exactly as it appears in this diagram.

1. Work with a partner to try and recreate this graph on your calculator screen. Make sure you pay attention to the height of the jumper at each oscillation, as given in the table.




| ***hor*** | ***vert*** | ***distance from midline*** |
|---|---|---|
| 0 | **120** | 40 |
| 1 | 48 | 32 |
| 2 | **105.6** | 25.6 |
| 3 | 59.52 | 20.48 |
| 4 | **96.38** | 16.38 |
| 5 | 66.89 | 13.11 |
| 6 | **90.49** | 10.49 |
| 7 | 71.61 | 8.39 |
| 8 | **86.71** | 6.71 |
| 9 | 74.63 | 5.37 |
| 10 | **84.30** | 4.30 |
| 11 | 76.56 | 3.44 |
| 12 | **82.75** | 2.75 |
| 13 | 77.80 | 2.20 |
| 14 | **81.76** | 1.76 |
| 15 | 78.59 | 1.41 |
| 16 | **81.13** | 1.13 |
| ⋮ | ⋮ | |

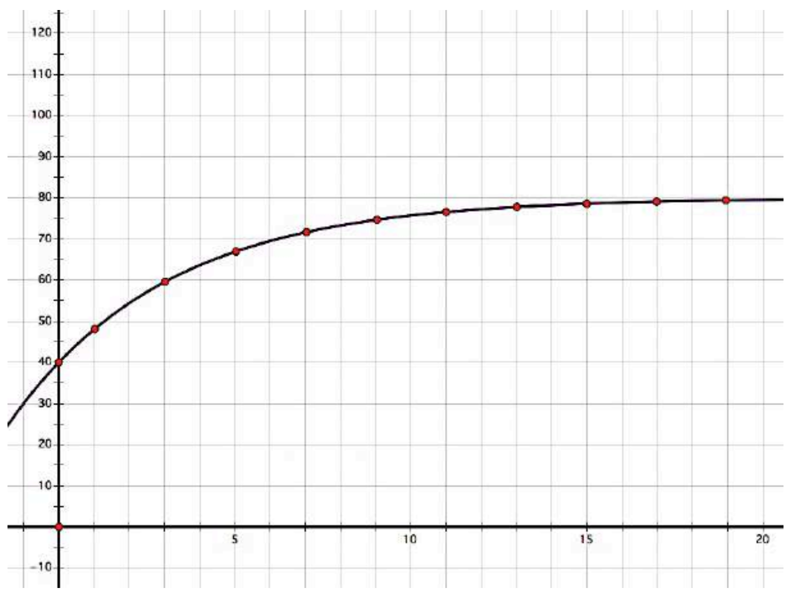
Record your equation of this graph here:

After a thrilling ride on the *Bungee Jump Simulator*, you are met by your host who has a new puzzle for you. "As you are aware," says the engineer, "temperatures around here are very cold at night, but very warm during the day. When designing rides we have to take into account how the metal frames and cables might heat up throughout the day. Our calculations are based on Newton's Law of Heating. Newton found that while the temperature of a cold object increases when the air is warmer than the object, the rate of change of the temperature slows down as the temperature of the object gets closer to the temperature of its surrounding."

Of course the engineer has a graph of this situation, which he says "represents the decay of the difference between the temperature of the cables and the surrounding air."

Your friends think this graph reminds them of the points at the bottom of each of the oscillations of the bungee jump graph.

2. Using the clue given by the engineer, "This graph represents the *decay* of the difference between the temperature of the cables and the surrounding air," try to recreate this graph on your calculator screen. (Hint: What types of graphs do you generally think of when you are trying to model a growth or decay situation? What transformations might make such a graph look like this one?)




Record your equation of this graph here:

# 8.3 The Bungee Jump Simulator – Teacher Notes

## *A Solidify Understanding Task*

**Purpose**: The purpose of this task is to solidify thinking about combining function types using such operations as addition or multiplication. In the previous task students have noticed how the characteristics of both types of functions are manifested in the resulting graphs. In this task they become more precise about how the functions combine by trying to determine the exact equation that will produce a given graph.

**Core Standards Focus**:

**F.BF.1b** Write a function that describes a relationship between two quantities.★

Combine standard function types using arithmetic operations. For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.

**Standards for Mathematical Practice:**
**SMP 4 – Model with mathematics**
**SMP 5 – Use appropriate tools strategically**
**SMP 6 – Attend to precision**

**Vocabulary:** If you have not previously done so, use the word *modeling* to describe the process of applying mathematics we know to solve real-world problems, usually requiring us to make and vary assumptions based on how well our mathematical model fits data from the context.

**The Teaching Cycle:**
**Launch (Whole Class):**
Engage students in the scenario of the task. Point out the two graphs in the task, and discuss what they are supposed to model according to the engineer in the story. Inform students that they are

not finished with the task until they can produce these exact graphs on their calculators and explain what functions they used and how they tweaked each function to make it fit the details of the given graphs.

**Explore (Small Group):**

For both of the graphs, encourage students to try something. They have experimented with similar graphs on the previous task, and guess and check can provide a lot of insights into these graphs. Students should try something and then refine their initial guesses. The following specific prompts may help.

If students are struggling with the first graph, help them deconstruct it into individual functions by asking questions such as, “What if the oscillations of the bungee jumper didn’t level off, but kept returning to the same height or distance from the ground as in the previous oscillation? What function would model that behavior?” Or, “What if we just examine the points at the top of each oscillation of the graph? What kind of function would pass through those points?” These points are listed in boldfaced type in the table. The column on the table listed as ‘distance from midline’ illustrates the decay of the amplitude of the cosine graph.

For the second graph, remind students that they have two hints—the engineer used the word “decay” in describing the graph, and one of their friends said the graph looked like the points at the bottom of each of the oscillations of the bungee jump graph. Ask students how these points are related to the points at the top of each oscillation, which we had to consider in the previous graph. Perhaps having students draw the midline of the bungee jump graph will help them see the reflection across this midline of the graph that goes through the maximum points and the graph that goes through the minimum points. If needed, ask students if they can think of a series of transformations that could take an exponential decay function and make it look like this graph that is approaching the midline from below.

**Discuss (Whole Class):**

Have students who have successfully duplicated the graphs describe their work. If no one has been successful, focus on the first graph and have students create an equation for the undampened, sinusoidal graph, $y = 40\cos(\pi x) + 80$. Then have them create an equation for the exponential decay graph that would fit the points at the top of each curve, $y = 40 \cdot (0.8)^x + 80$. This is an exponential decay function that has been shifted up to the height of the midline. The exponential decay function $y = 40 \cdot (0.8)^x$ can be found from the data giving the distance from the midline. Finally, have students consider how they might combine the two functions. The important issue here is to note that functions of the form $y = a\sin(bx) + c$ would oscillate around the midline at $y = c$ with an amplitude given by $a$. If we replace the parameter $a$ with an exponential decay function, we can cause the amplitude to dampen down to 0, just as the exponential decay function approaches 0. Note that we don't need to include the vertical shift twice. It is as if we had created the entire graph to oscillate and decay around the *x*-axis, and then shifted this whole behavior up to the location of the midline. It might be helpful to provide students with a copy of the graph on the following page as they consider how the exponential decay function describes the decay of the amplitude away from the midline.

If there is time you can also work through the second graph using the hints as discussed in the explore section. The important issue here is to focus on the exponential decay function used in the previous graph—the version that decays to the *x*-axis. If this function is reflected over the *x*-axis and then shifted up to the midline, it would pass through the minimum points of the bungee jump graph, which are the same points as in the heating up graph.

**Aligned Ready, Set, Go: Modeling with Functions 8.3**

READY, SET, GO! Name Period Date

## READY

Topic: Evaluating functions

**Evaluate each function as indicated. Simplify your answers when possible. State *undefined* when applicable.**

1. $f(x) = x^2 - 8x$

   *a)* $f(0)$ *b)* $f(-10)$ *c)* $f(5)$ *d)* $f(8)$ *e)* $f(x+2)$

2. $g(x) = \frac{3x-5}{x}$

   *a)* $g(-1)$ *b)* $g(10)$ *c)* $g\left(\frac{1}{3}\right)$ *d)* $g(0)$ *e)* $g(2x+4)$

3. $h(x) = sin(x)$

   *a)* $h(\pi)$ *b)* $h\left(\frac{3\pi}{2}\right)$ *c)* $h\left(\frac{11\pi}{6}\right)$ *d)* $h\left(\frac{5\pi}{4}\right)$ *e)* $h\left(\cos^{-1}\left(\frac{-1}{2}\right),\ x < \pi\right)$

4. $w(x) = tan(x)$

   *a)* $w(\pi)$ *b)* $w\left(\frac{3\pi}{2}\right)$ *c)* $h\left(\frac{7\pi}{6}\right)$ *d)* $h\left(\frac{3\pi}{4}\right)$ *e)* $h\left(\cos^{-1}\left(\frac{-1}{2}\right),\ x < \pi\right)$

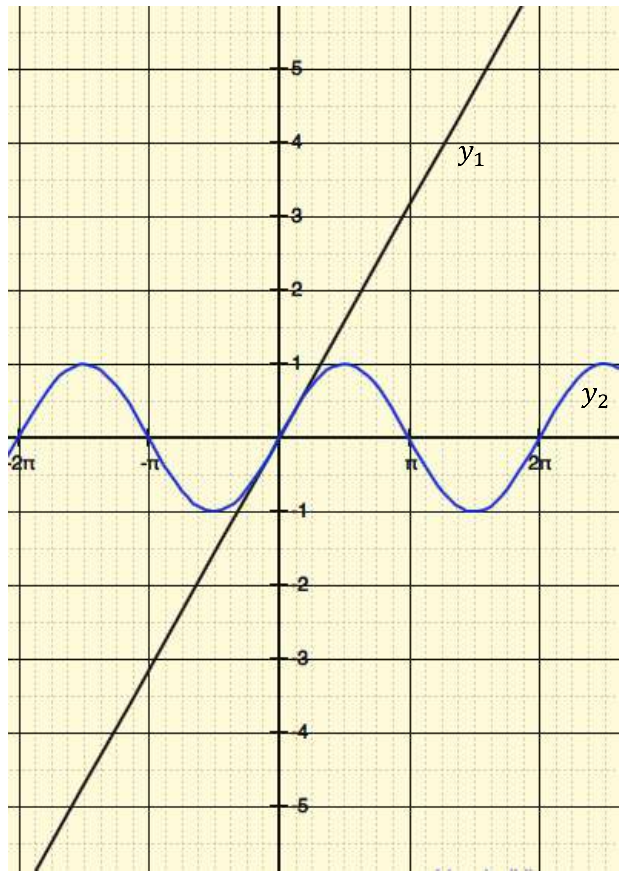
## SET

Topic: Damping functions

Two functions are graphed. Graph a third function by multiplying the two functions together. Use the table of values to assist you. It may help you to change the function values to decimals.

5.

| $x$ | $y_1 = x$ | $y_2 = sinx$ | $y_3 = (x)sinx$ |
|---|---|---|---|
| $-2\pi$ | | | |
| $-\frac{3\pi}{2}$ | | | |
| $-\pi$ | | | |
| $-\frac{\pi}{2}$ | | | |
| 0 | | | |
| $\frac{\pi}{2}$ | | | |
| $\pi$ | | | |
| $\frac{3\pi}{2}$ | | | |
| $2\pi$ | | | |




6. After you have graphed $y_3$, graph the line $y_4 = -x$. What do you notice about the graph of $y_3$ in relation to the graphs of $y_1$ *and* $y_4$?

## GO

Topic: Comparing measures of central tendency (mean, median, and mode)

During salary negotiations for teacher pay in a rural community, the local newspaper headlines announced: **Greedy Teachers Demand More Pay!** The article went on to report that teachers were asking for a pay hike even though district employees, including teachers, were paid an average of $70,000.00 per year, while the average annual income for the community was calculated to be $55,000 per household. The 65 schoolteachers in the district responded by declaring that the newspaper was spreading false information.

**Use the table below to explore the validity of the newspaper report.**

| Job Description | Number having job | Annual Salary |
|---|---|---|
| Superintendent | 1 | $258,000 |
| Business Administrator | 1 | $250,000 |
| Financial Officer | 1 | $205,000 |
| Transportation Coordinator | 1 | $185,000 |
| District secretaries | 5 | $ 55,000 |
| School Principals | 5 | $200,000 |
| Assistant Principals | 5 | $175,000 |
| Guidance Counselors | 10 | $ 85,000 |
| School Nurse | 5 | $ 83,000 |
| School Secretaries | 10 | $ 45,000 |
| Teachers | 65 | $ 48,000 |
| Custodians | 10 | $ 40,000 |

7. Which measure of central tendency (mean, median, mode) do you think the newspaper used to report the teachers' salaries? Justify your answer.

8. Which measure of central tendency do you think the teachers would use to support their argument? Justify your answer.

9. Which measure gives the clearest picture of the salary structure in the district? Justify.

10. Make up a headline for the newspaper that would be more accurate.

# 8.4 Composing and Decomposing

## *A Develop Understanding Task*



As the day at the amusement park get warmer, you and your friends decide to cool off by taking a ride on the *Turbulent Waters Dive*. As you are waiting in line your tour guide explains the mathematics behind designing the waiting area for a ride.

"As you can see," says the engineer, "the waiting area can be enlarged or reduced by moving a few chains around. The area we need for waiting guests depends on the time of day. We collect data for each ride so we can use functions to model the typical wait time and how much waiting area we need to provide for our guests."

And of course, your guide has the functions that represent this particular ride.

- **Average number of people in the *TWD* line as a function of time:** $\boxed{p(t) = 3000\cos\left(\tfrac{1}{5}(t-3)\right)}$

  → ***t*** is the number of hours before or after noon, so $t = 2$ represents 2:00 p.m. and $t = -2$ represents 10:00 a.m.

  → ***p*** represents the number of people in line

- **Waiting area required as a function of the number of people in line:** $\boxed{a(p) = 4p + 100}$

  → ***a***, the waiting area, is measured in square feet

- **Wait time for a guest as a function of the number of people in line:** $\boxed{W(p) = 60 \cdot \left(\dfrac{p-1500}{1500}\right)}$

  → ***W***, the wait time, is measured in minutes

1. How much waiting area is required for the guests in line for the *Turbulent Waters Dive* at each of the times listed in the following table?

| ***Time of Day*** | ***Waiting Area Required* (sq. ft.)** |
|---|---|
| 10:00 a.m. | |
| 12:00 noon | |
| 2:00 p.m. | |
| 4:00 p.m. | |
| 8:00 p.m. | |

   a. For each instant in time you had to complete a series of calculations. Describe how you found the waiting area at different times.

   b. Can you create a single rule that will determine the waiting area as a function of the time of day?

2. What is the wait time for a guest that arrives at the end of the line for the *Turbulent Waters Dive* at each of the times listed in the following table?

| ***Time of Day*** | ***Wait Time* (minutes)** |
|---|---|
| 10:00 a.m. | |
| 12:00 noon | |
| 2:00 p.m. | |
| 4:00 p.m. | |
| 8:00 p.m. | |

   a. For each instant in time you had to complete a series of calculations. Describe how you found the wait time at different times of the day.

b. Can you create a single rule that will determine the wait time as a function of the time of day?

To maintain crowd control when the lines get long, cast members dressed as pirates (the *Turbulent Waters Dive* has a pirate theme) mingle with the waiting guests. Their antics distract the guests who listen attentively to their pirate jokes. The number of cast members needed depends on the number of people waiting in the line.

• **Number of cast members needed as a function of the number of people in line:** $\boxed{c(t) = \frac{p}{150}}$

$\rightarrow$ ***p*** represents the number of people in line

$\rightarrow$ ***c*** represents the number of cast members needed

3. How many cast members are needed to entertain and distract the waiting guests at each of the following times of the day?

| ***Time of Day*** | ***Cast Members Needed*** |
|---|---|
| 10:00 a.m. | |
| 12:00 noon | |
| 2:00 p.m. | |
| 4:00 p.m. | |
| 8:00 p.m. | |
| *t* hours before or after noon<br>($t < 0$ before noon, $t > 0$ after noon) | |

On warm, sunny days misters are used to cool down the waiting guests. The number of misters that need to be turned on depends on the size of the waiting area that has been opened up to contain the number of people in line.

• **Number of misters needed as a function of the waiting area:**

$$\boxed{m(t) = \frac{a}{1000}}$$

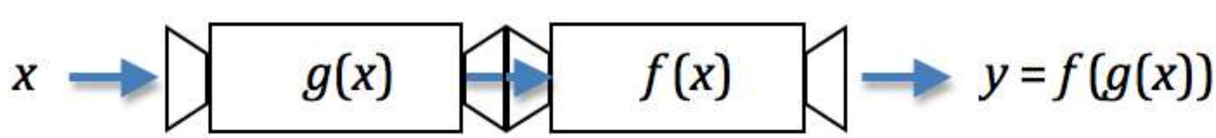
$\rightarrow$ ***a***, the waiting area, is measured in square feet

$\rightarrow$ ***m*** represents the number of misters to be turned on

4. How many misters need to be turned on to cool the waiting guests at each of the following times of day?

| ***Time of Day*** | ***Misters Needed*** |
|---|---|
| 10:00 a.m. | |
| 12:00 noon | |
| 2:00 p.m. | |
| 4:00 p.m. | |
| 8:00 p.m. | |
| $t$ hours before or after noon ($t < 0$ before noon, $t > 0$ after noon) | |

5. Explain how the following diagram might help you think about the work you have been doing on the previous problems. How does the notation used in the diagram support the way you have been combining functions in this task? This way of combining functions is called ***function composition***.



***Interpreting the Functions***

6. At what time of day is the number of people in line the largest?

7. What is the maximum number of people in line, based on the function for the average number of people in line?

8. When do you think the amusement park opens and closes, based on this function?

9. In terms of the story context, what do you think the 4 and the 100 represent in function rule for waiting area, $a(p) = 4p + 100$?

10. In terms of the story context, what might be the meaning of the 1500 in the function rule for wait time, $w(p) = 60 \cdot \left(\frac{p-1500}{1500}\right)$?

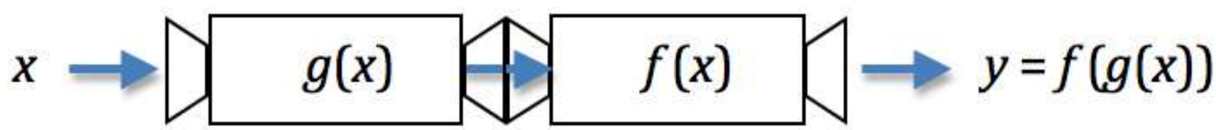
11. In terms of the story context, what might be the meaning of the 150 in the function rule for cast members needed, $c(t) = \frac{p}{150}$?

12. In terms of the story context, what might be the meaning of the 1000 in the function rule for the number of misters needed, $m(t) = \frac{a}{1000}$?

# 8.4 Composing and Decomposing – Teacher Notes

*A Develop Understanding Task*

**Purpose**: The purpose of this task is to consider another way that functions can be combined—by function composition. When functions are added or multiplied, it is the outputs of the individual functions for a particular input that are added or multiplied together. In function composition, the output of one function becomes the input of the other. This is represented algebraically by writing one function inside the other: $y = f(g(x))$. Note that to find the output value for a particular input value $x$, we would first evaluate $g(x)$, and then use this resulting value as the input for function $f$. The following diagram illustrates the idea of a new function relationship being created by the composition of two other functions:




**Core Standards Focus**:

**F.BF.1c** Write a function that describes a relationship between two quantities.★

(+) Compose functions. For example, if T(y) is the temperature in the atmosphere as a function of height, and h(t) is the height of a weather balloon as a function of time, then T(h(t)) is the temperature at the location of the weather balloon as a function of time.

**Standards for Mathematical Practice:**

**SMP 4 – Model with mathematics**

**Vocabulary:** This task introduces students to *function composition*.

**The Teaching Cycle:**

**Launch (Whole Class):**

Introduce the scenario for this task by asking students if they have ever waited in line for an amusement park ride or another type of event, and if the vendor made any effort to manage the wait time. Students have probably observed lines that weave back and forth in parallel rows until you get to the front of the line. Perhaps students have observed how lines can be adjusted by roping off certain areas or by opening up new ones. Perhaps they have been on rides where the waiting area provided a form of entertainment itself.

Long lines, crowds, and the hot sun contribute in negative ways to the experience of a guest on a ride. Therefore, amusement parks collect data in order to determine ways to minimize the discomfort of the guests. This task deals with some of the mathematics of waiting in lines.

Discuss the purpose of the first three functions listed in the task, and the quantities represented by the input and output variables.

Note: The section at the end of the task, *Interpreting the Functions*, could be used as part of the launch if you want students to understand more of the details of what the constants and coefficients in each function represent. This section is left for last to allow students to get into the work of composing functions more quickly, and to give fast finishers significant work to do after they complete the main portion of the task. Feel free to use it as part of the launch (or to relaunch the task) if you feel it is needed to help students work with each function.

**Explore (Small Group):**

There are two approaches to working on these problems: we can decompose the scenario into its component functions and work with them individually and in sequence, or we can compose a single function that takes care of the sequence of computations. It is assumed that students will begin by working through a two-step process as they fill out the table on question 1: first, calculate the number of people in line as a function of the given time; second, calculate the area of the waiting region designated for the guests waiting in line. Watch for students who begin to shortcut this process by thinking of a single function that takes care of this sequence of computational steps, as prompted by parts a and b.

Listen for students who may create the composition function first, before filling out the tables for questions 2 and 3.

Listen for how students tackle question 4. Do they go back to decomposing the scenario into a three-step process: first, calculate the number of people in line as a function of time; second, calculate the waiting area as a function of the number of people; third, calculate the number of misters needed as a function of the area? Or, do they use the outputs from their work on question 1 as inputs to complete the table in question 4? Or, do they create a single function that takes care of the entire sequence of computations?

The section, *Interpreting the Functions*, is for students who may move quickly through the task and need some additional significant work to do. Identify students who have discussed this portion of the task and can assign a meaning to each of the parameters in each function rule.

**Discuss (Whole Class):**

Begin by discussing questions 6-12, to give students a deeper understanding of the functions they are working with and the stories they tell. Use student explanations of the parameters in the functions as much as possible. For example, the function that represents the number of people in line suggests that at 3:00 p.m. the ride has the maximum number of people waiting in line. This maximum number is 3000, the amplitude of the cosine graph. Since this is a water ride, with the potential of getting wet, the ride would be more popular during the hot afternoon than earlier in the morning or after sunset. This accounts for the $(t - 3)$ in the function rule since the cosine has a maximum value when the input is 0, and this expression makes the input 0 when $t = 3$. Discussing this idea explicitly gives students insight into why the expression $t - 3$ inside the cosine function shifts the cosine graph to the right. Examining the graph of this function suggests that the park opens at 7:00 a.m. and closes at 11:00 p.m., since those are the hours between which the function gives a positive value for the number of people waiting in the line.

For question 9 students should identify that the 4 represents 4 square feet per person waiting in line, and that the 100 represents a starting area of 100 square feet, which might be accounted for as the loading area of the ride.

It may be more difficult for students to determine the meaning of the 1500 in the function for wait time, as requested in question 10. The 1500 gives the hourly capacity of the ride. There are only people *waiting* in line (instead of moving through the line) when $p > 1500$. For example, if $p = 2000$, there are 500 more people in line than can freely move through the ride without waiting. $\frac{500 \ people}{1500 \ people/hour} = \frac{1}{3} \ hour$, which we convert to minutes by multiplying by 60.

After discussing the details of the meaning of each of these three functions, ask students to use a diagram similar to that given in question 5 to describe their work on questions 1-4. It may be interesting, if you have time, to have students determine a reasonable domain for each of the composite functions based on the actual story context. For example, would it make sense to open up more waiting area if the number of people lining up during a particular hour is less that 1500, since that number of people can move freely through the line without waiting?

Discuss the notation used for function composition $y = f(g(x))$ from the diagram. You may choose to introduce an alternative notation for function composition: $(f \circ g)(x) = f(g(x))$. (Note that in this notation $g$ is the "inside" function, the function that is evaluated first, and then the result of that evaluation is used as the input into function $f$.) Make sure that students can match the diagram to the notation used for function composition.

**Aligned Ready, Set, Go: Modeling with Functions 8.4**

READY, SET, GO! Name Period Date

## READY

Topic: Recognizing the order of operations in a composite function

**Each expression contains 2 operations. One of the operations will be "inside" the second operation. Identify the "inside" operation as $u$ by writing $u$ = ________. Then substitute $u$ into the expression so that the "outside" operation is being performed on $u$.**

**Example:** $\boldsymbol{Given}$: $\mathbf{5x^3}$.

**I can see two operations on x. First the x is being cubed and then $x^3$ is multiplied by 5. Therefore, if $u = x^3$, then $5x^3 = 5u$.**

1. Would the answer in the example have been different if you were given $(5x)^3$? Explain

2. $(x-6)^2$

$u$ =

3. $\tan(x+4)$

$u$ =

4. $\sqrt[3]{(2x-7)}$

$u$ =

5. $-9(x+5)$

$u$ =

6. $\dfrac{5}{x^2}$

$u$ =

7. $(\sin x)^4$

$u$ =

## SET

Topic: Creating formulas for composite functions

Recall that $\boldsymbol{f(g(x)) = (f \circ g)(x)}$.

8. Let $f(x) = 2x^2 - 4$ *and* $g(x) = 5x$. Find each and simplify.

   a) $(f \circ g)(1)$ b) $(g \circ f)(1)$ c) $(f \circ f)(-2)$ d) $(g \circ g)(-1)$

9. Let $f(x) = \frac{8}{x-3}$ *and* $g(x) = \frac{15}{x+1}$. Find each and simplify.

   a) $(f \circ g)(x)$ b) $(g \circ f)(x)$ c) $(f \circ f)(x)$ d) $(g \circ g)(x)$

10. Use your answers for a) and b) in problem 9 to calculate the two problems below.

    a) $(f \circ g)(-1)$ b) $(g \circ f)(3)$

11. Now use $f(x) = \frac{8}{x-3}$ *and* $g(x) = \frac{15}{x+1}$ to calculate $(f \circ g)(-1)$ and $(g \circ f)(3)$.

    a) Describe the problem that you encountered when calculating *f(x)* and *g(x)* separately.

    b) Do you think that the answer you derived in #10 is valid based on what happened in #11? Justify your answer.

12. Describe the domains for a) $(f \circ g)(x)$ b) $(g \circ f)(x)$ c) $(f \circ f)(x)$ d) $(g \circ g)(x)$

13. What makes the domain for each composition different?

## GO

Topic: Writing equations of polynomials given the degree and the roots

**Write the equation of the polynomial with the given features.**

| | Degree of polynomial | Given roots (you may have to determine others): | Leading coefficient | Equation in standard form: |
|---|---|---|---|---|
| 14. | 3 | -2, 1, and -1 | 3 | |
| 15. | 4 | $(2+i), 4, 0$ | 1 | |
| 16. | 5 | 1 multiplicity 2, -1 multiplicity 2, and 3 | -1 | |
| 17. | 4 | $(3-i), \sqrt{2}$ | -2 | |

# 8.5 Translating My Composition

*A Solidify Understanding Task*



All this work with modeling rides and waiting lines at the local amusement park may have you wondering about the variety of ways of combining functions. In this task we continue building new functions from old, familiar ones.

Suppose you have the following "starter set" of functions.

$f(x) = x + 5$

$g(x) = x^2$

$h(x) = 3x$

$j(x) = 2^x$

$k(x) = x - 1$

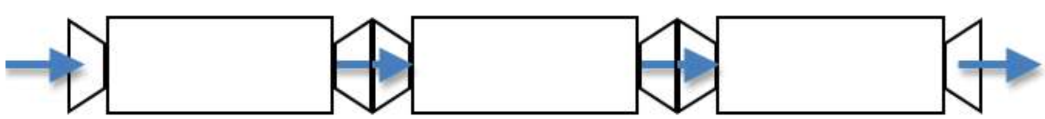
You and a partner will then do the following steps with your given set of functions:

1st. Build a composite function using any three of the above function rules in any order

2nd. Write your final function rule as a single algebraic expression in terms of $x$

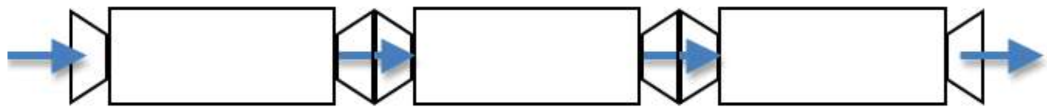
3rd. Give your function rule to your partner, you should also receive a function rule from your partner

4th. Your partner should fill in the following diagram, decomposing your rule into its component parts and combining them in the correct order



1. First, let's try this example:

   Your partner gives you $f_1(x) = 3(x+5)^2$. Complete this diagram to decompose this composition into its component parts.

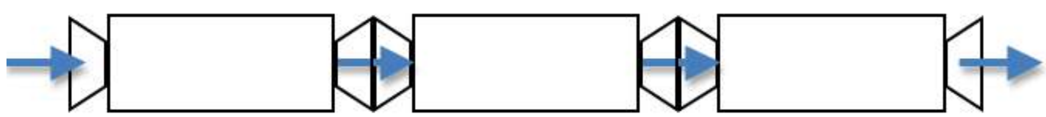


2. To test your decomposition you can try running a number or two through your chain of function machines, and see if you get the same results as when you evaluate the function rule for the same numbers. What do you notice when you do this?

3. Now it's your turn! Create your own function rule using the set of functions given at the beginning of this task and following the four steps given above. Your partner should do the same and give you his or her function rule.

   Record the function rule you received here:

   Complete this diagram to decompose your partner's composition into its component parts.



   Test your decomposition for a few values. Make any adjustments necessary based on your test results.

4. Instead of giving you the function rule, suppose your partner gives you the following input-output table. Can you create the composition function rule based on this information? Describe how you used the numbers in this table to create your rule.

| $x$ | $f(x)$ |
|---|---|
| 0 | 5 ½ |
| 1 | 6 |
| 2 | 7 |
| 3 | 9 |
| 4 | 13 |
| 5 | 21 |

5. Is function composition commutative? Give reasons to support your answer.

# 8.5 Translating My Composition – Teacher Notes

## *A Solidify Understanding Task*

**Purpose**: The purpose of this task is to extend students understanding of composition of functions by considering the roles of the individual functions of which each composition function is composed. Students will observe that function composition is not commutative. They will also notice that functions created by composition can often be interpreted as transformations of standard (or "parent") functions.

**Core Standards Focus**:

**F.BF.1c** Write a function that describes a relationship between two quantities.★

(+) Compose functions. For example, if T(y) is the temperature in the atmosphere as a function of height, and h(t) is the height of a weather balloon as a function of time, then T(h(t)) is the temperature at the location of the weather balloon as a function of time.

**F.BF.3** Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them.

**Secondary Math III note:** *For F.BF.3, note the effect of multiple transformations on a single function and the common effect of each transformation across function types. Include functions defined only by a graph. Use transformations of functions to find more optimum models as students consider increasingly more complex situations.*

**Standards for Mathematical Practice:**

**SMP 2 – Reason abstractly and quantitatively**

**Vocabulary:** This task describes the *decomposition of functions,* which allows one to think of a composite function in terms of its component parts.

**The Teaching Cycle:**

**Launch (Whole Class):**

Set up the example problem on the first page of the task by pointing out the "starter set" of functions and the four rules for creating more complex functions by composition. Give students time to fill in the three functions in the diagram in the order they think replicates the given composition function. Have a student suggest his or her selection and order of functions, and then see if there are different orderings (or perhaps a different set of functions) that were recorded by other students in the class. If so, post them also. Then have students test out the suggested function(s) using a few test values, as suggested by question 2. Finish the launch by asking students to write the composition notation, $h(g(f(x)))$, and verify algebraically that they have decomposed the given function in the correct order. Tell students that they are now going to construct their own functions using this same "starter set" of functions and the same four steps. They are to trade their composition with their partner and try to decompose their partner's function. If partner's disagree, they should work to resolve their disagreement.

**Explore (Small Group):**

You may assign students to generate as many unique functions from this "starter set" as you want—there are 60 different possible combinations. For a variation, you might extend the number of functions used to four, or even all five. You can also allow repetition of functions, such as, $g(f(g(x)))$. Listen for students who are making the connection between the order of function composition and the order of operations when the given expression is evaluated for a given value of *x.* For example, to evaluate the expression $3(x+5)^2$ for *x* = 2, the order of operations would first tell us to *add 5* to the 2, then *square the result*, then *multiply by 3*. This order of operations for evaluating the expression also defines the sequence of compositions that created this expression.

One way students might approach question 4 is to create a column of first differences of the output values.

| *x* | *f(x)* | Δ*f(x)* |
|---|---|---|
| 0 | 5 ½ | |
| | | ½ |
| 1 | 6 | |
| | | 1 |
| 2 | 7 | |
| | | 2 |
| 3 | 9 | |
| | | 4 |
| 4 | 13 | |
| | | 8 |
| 5 | 21 | |

Note that the first differences suggest that this is an exponential function, but the doubling behavior has been masked somewhat by other transformations. Students may notice that if they subtract 5 from each of the *f(x)* values, they get a sequence of numbers that highlights the successive doubling of the outputs. The results of subtracting 5 are still off by 1 position from the sequence that would be generated by $2^x$.

**Discuss (Whole Class):**

The following three issues should come up in the discussion:

1. The order of operations when evaluating a composite function for a particular value of *x* also defines the order of the composition sequence.

2. Function composition is not commutative.

3. One way to explain the counterintuitive notion of horizontal transformations [e.g., the graph of $f(x-1)$ is shifted 1 unit to the *right* of the graph of $f(x)$] is to examine the shifts in the *x*-values for corresponding tables. For example,

$f_1(x) = 2^x + 5$

| $x$ | | -1 | 0 | 1 | 2 | 3 | 4 | |
|---|---|---|---|---|---|---|---|---|
| $f_1(x)$ | | 5 ½ | 6 | 7 | 9 | 13 | 21 | |

$f_2(x) = 2^{x-1} + 5$

| $x$ | -1 | 0 | 1 | 2 | 3 | 4 | | |
|---|---|---|---|---|---|---|---|---|
| $f_1(x)$ | | 5 ½ | 6 | 7 | 9 | 13 | 21 | |

| $x$ | | -1 | 0 | 1 | 2 | 3 | 4 | |
|---|---|---|---|---|---|---|---|---|
| $f_1(x)$ | | | 5 ½ | 6 | 7 | 9 | 13 | 21 |

Note that the *input* values have all been shifted one unit to the left, or the *output* values have all been shifted one unit to the right.

**Aligned Ready, Set, Go: Modeling with Functions 8.5**

**READY, SET, GO!** Name Period Date

## READY

Topic: Using a table to find the value of a composite function

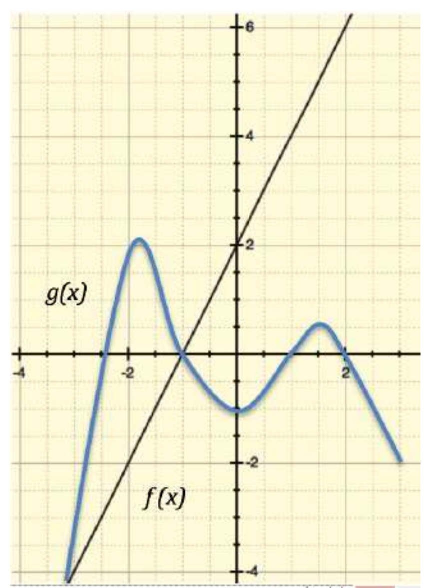
**Use the table to find the indicated function values.**

| x | $f(x)$ | $g(x)$ |
|---|---|---|
| -2 | 2 | 3 |
| -1 | 1 | -2 |
| 0 | 3 | -24 |
| 1 | -1 | -1 |
| 2 | 0 | -8 |
| 3 | 19 | 0 |

1. $f(g(3))$
2. $f(g(1))$
3. $g(f(-2))$
4. $g(f(-1))$
5. $g(f(0))$
6. $g(g(-2))$
7. $f(f(0))$
8. $g(f(1))$

9. Do the graphs of $f(x)$ *and* $g(x)$ described in the table ever intersect each other?

   How do you know?




**Use the graph to find the indicated values.**

10. $f(g(-2))$
11. $f(g(-1))$
12. $f(g(1.5))$
13. $f(f(0))$

**Need help?** Visit **www.rsgsupport.org**

## SET

Topic: Creating a composite function given its components

Let $f(x) = x^2, g(x) = 5x, and\ h(x) = \sqrt{x} + 2.$

**Express each function as a composite of $\boldsymbol{f, g, and/or\ h}$.**

14. $F(x) = x^4$

15. $G(x) = 5x^2$

16. $P(x) = x + 2$

17. $R(x) = 5\sqrt{x} + 10$

18. $Q(x) = 25x$

19. $H(x) = 25x^2$

20. $D(x) = \sqrt{\sqrt{(x)} + 2} + 2$

21. $B(x) = x + 4\sqrt{x} + 4$

22. $K(x) = \sqrt{5x} + 2$

## GO

Topic: Finding the zeros of a polynomial

**Solve for all of the values of x. Identify any restrictions on x.**

23. $x^2 + 6 = 5x$

24. $5x^3 = 45x$

25. $x^4 - 26x^2 + 25 = 0$

26. $1 + \frac{1}{x} = \frac{12}{x^2}$

27. $\frac{x}{6} - \frac{1}{2} - \frac{3}{x} = 0$

28. $\frac{1}{x^2} = 9$

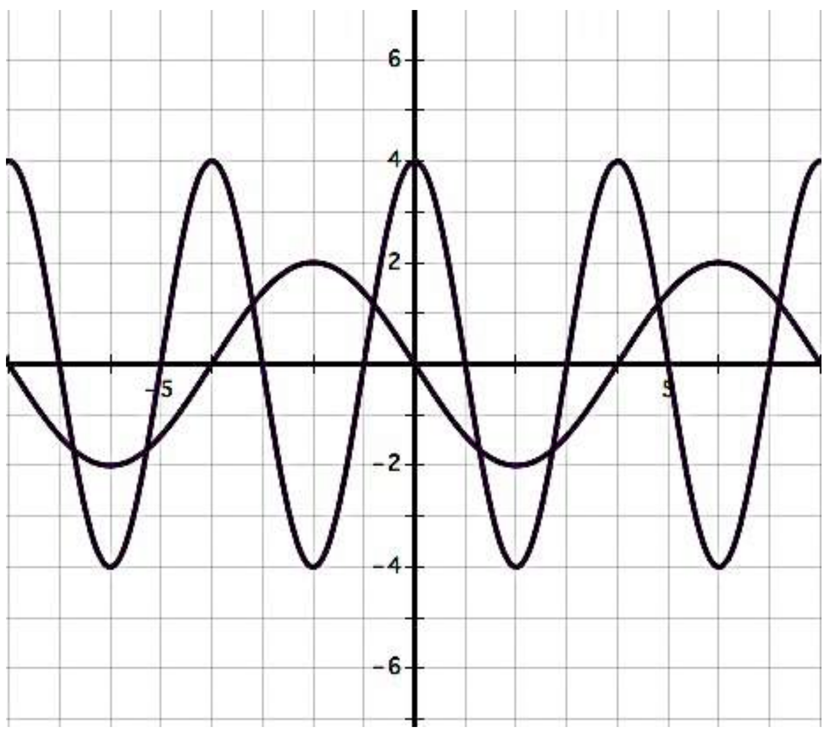
# 8.6 Different Combinations

## *A Practice Understanding Task*



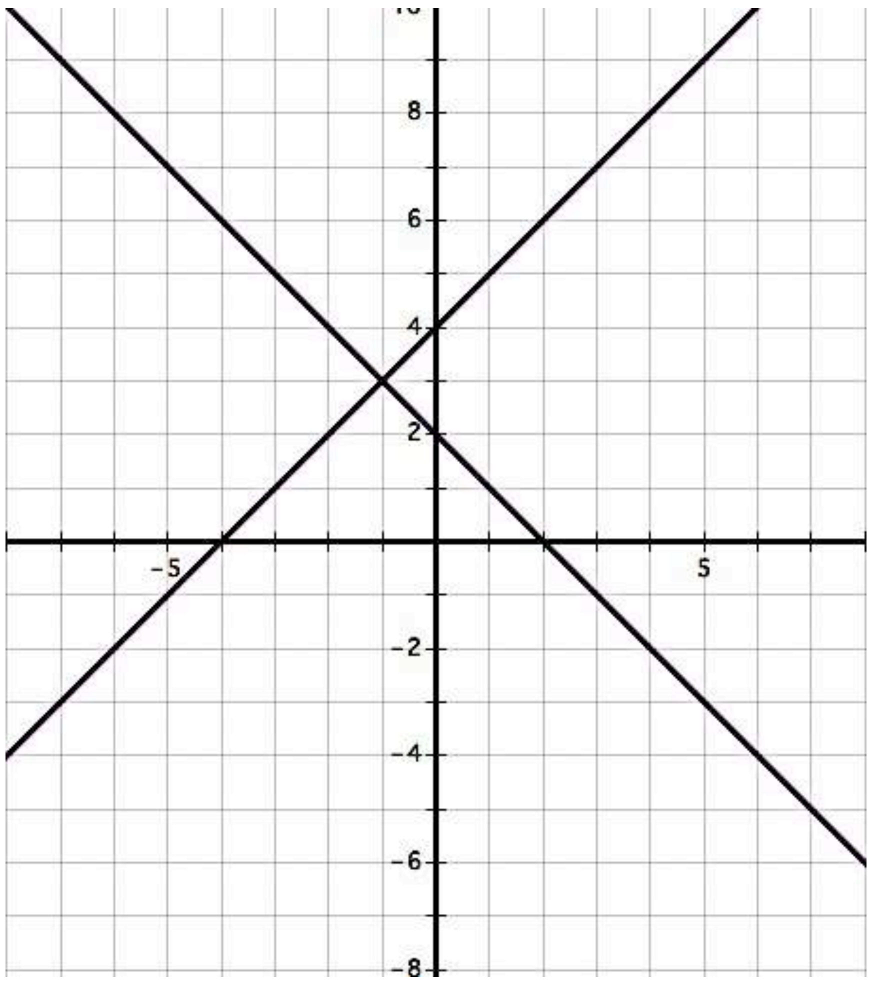
We have found the value of being able to combine different function types in various ways to model a variety of situations. In this task you will practice combining functions when they are described in different ways: graphically, numerically or algebraically.

1. Add the following two functions together graphically. That is, do not write the algebraic rules for each individual function, add them together, and then graph the result. See if you can produce the resulting graph by just working with the points on the two graphs and considering what happens when two functions are combined using the operation of addition.



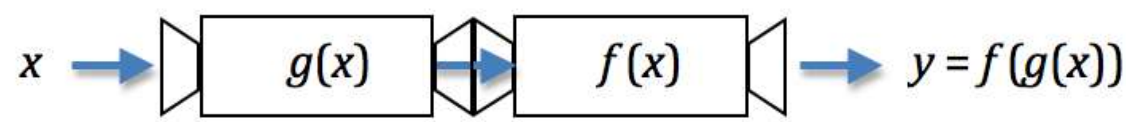

Which points are most helpful in determining the shape of the resulting graph, and why?

2. Multiply the following two functions together graphically. That is, do not write the algebraic rules for each individual function, multiply them together, and then graph the result. See if you can produce the resulting graph by just working with the points on the two graphs and considering what happens when two functions are combined using the operation of multiplication.




Which points are most helpful in determining the shape of the resulting graph, and why?

3. In a previous task we used the following diagram to illustrate function composition. Draw a similar type of diagram to illustrate what happens when two functions are combined by addition or multiplication. Your diagram should clearly show how the output values are obtained for specific input values.




4. Functions $f$ and $g$ are defined numerically in the following table. No other points exist for these functions other than the points given. Find the output values for each of the other combinations of functions indicated. Fill in as many points as are defined based on the give data. Use the same input values for all functions.

| $x$ | $f(x)$ | $g(x)$ | $(f+g)(x)$ | $f^{-1}(x)$ | $g(f(x))$ | $f(g(x))$ |
|---|---|---|---|---|---|---|
| **0** | **0** | **-3** | | | | |
| **1** | **2** | **-2** | | | | |
| **2** | **4** | **-1** | | | | |
| **3** | **6** | **0** | | | | |
| **4** | **8** | **1** | | | | |
| **5** | **10** | **2** | | | | |
| **6** | **12** | **3** | | | | |
| **7** | **14** | **4** | | | | |
| **8** | **16** | **5** | | | | |

5. Remember the race between the tortoise and the hare? Well, their friends and families have come to cheer them on, and have positioned themselves at various places along the course. Because rabbits are quick and eager to know the outcome of the race, more of them have congregated towards the end of the course. Because turtles are slow and more anxious to cheer their champion off to a good start, more of them have congregated at the beginning of the race. In fact, the density (or amount of animals/meter) of turtles and rabbits along the course as a function of the distance from the starting line is given by the following functions.

   ***The tortoise:*** $a(d) = 243 \cdot \left(\frac{1}{3}\right)^{\frac{1}{20}d}$ ($a$ is in turtles per meter, $d$ in meters)

   ***The hare:*** $a(d) = 2^{\frac{1}{10}d}$ ($a$ is in rabbits per meter, $d$ in meters)

   The distance from the starting line, as a function of the elapsed time since the start of the race, is given for the tortoise and the hare by the following functions.

   ***The tortoise:*** $d(t) = 2^t$ ($d$ in meters, $t$ in seconds)

   ***The hare:*** $d(t) = t^2$ ($d$ in meters, $t$ in seconds)

The tortoise and the hare are anxious to know how many of their friends and family they are passing at any instant in time along the race.

- Create functions for the tortoise and for the hare that will calculate the number of turtles or rabbits they will pass at any time, $t$, after the race begins. Include a reasonable domain for each function.

- If the race is 100 meters long, create a function that will tell how many spectators, rabbits and turtles, are watching at any distance away from the start of the race?

- Who is passing the most friends and families, the tortoise or the hare, 5 seconds after the race began?

# 8.6 Different Combinations – Teacher Notes

## *A Practice Understanding Task*

**Purpose**: The purpose of this task is for students to clarify and practice strategies for combining functions by addition, multiplication or composition. Students will combine functions that have been defined graphically, numerically and algebraically.

**Core Standards Focus**:

**F.BF.1b** Write a function that describes a relationship between two quantities.★

Combine standard function types using arithmetic operations. For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.

**F.BF.1c** Write a function that describes a relationship between two quantities.★

(+) Compose functions. For example, if T(y) is the temperature in the atmosphere as a function of height, and h(t) is the height of a weather balloon as a function of time, then T(h(t)) is the temperature at the location of the weather balloon as a function of time.

**Standards for Mathematical Practice:**

**SMP 2 – Reason abstractly and quantitatively**

**Vocabulary:** In this task students are asked to create a *"reasonable domain",* which is more generally referred to as a *restricted domain,* for a context.

**The Teaching Cycle:**

**Launch (Whole Class):**

Remind students that the work of this module has focused on combining various types of functions in a variety of ways: addition, subtraction, multiplication, division or composition. We have tried to find a way to describe the resulting function with a single algebraic rule, graph or table. By definition, a function produces a unique output for each input value in its domain. In this task students will think more clearly about how to determine the output for each input when the new function is a combination of functions that may be defined with graphs (see questions 1 and 2), with tables (see question 4), or with algebraic rules (see question 5).

**Explore (Small Group):**

For questions 1-3: Students have added or multiplied functions together before to produce graphs that illustrate the combined effects of both, but they have done so by first writing the equations of the individual components, adding or multiplying the algebraic rules together, and then verifying that the resulting function produced the desired graph. In this task they are asked not to write the algebraic equations of the component pieces, but to consider how they might generate point on the combined function's graph by just using the points on the graphs of the component functions. Listen for the ways they choose to do this. Do they recognize that they can select a particular value for $x$, find the corresponding $y$ values for each individual function, and then add (or multiply) those $y$ values together to produce the output of the combined function? To sketch an accurate graph of the combined function, students will need to produce several key points. Listen for how they determine which points to plot and how to think about the shape of the graph between plotted points. Do they just select random points, or do they recognize the importance of using points whose $y$ values are 0, or a maximum or minimum value on one of the graphs? Do they recognize the significance of points where the two graphs intersect? How do they think about connecting the points to create a continuous graph?

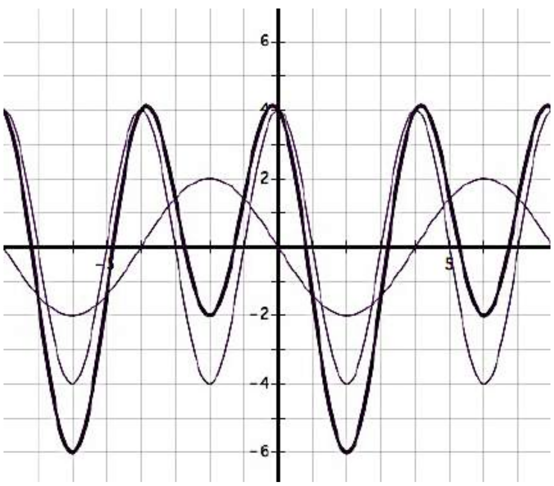
For question 4: Point out to students that functions $f$ and $g$ consist only of the order-pairs of coordinates given in the table, and they should not assume anything about the behavior between the points. This means that some values on the table will be undefined.

For question 5: Students will need to determine how to combine the functions for each different situation: by addition or by composition.

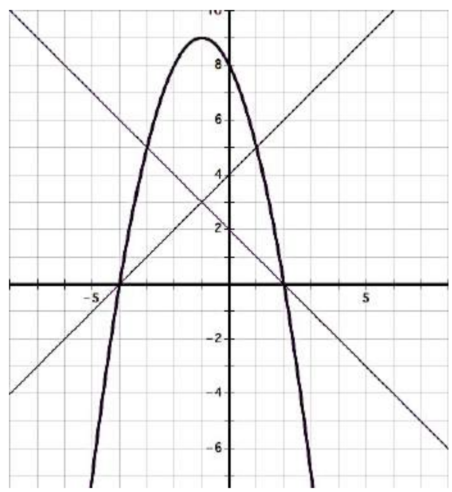
**Discuss (Whole Class):**

Here are the results you should expect from students.

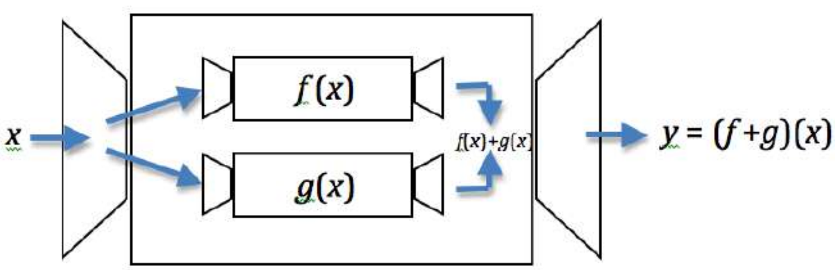
Question 1:




Question 2:




Question 3: (One possible diagram)



Question 4:

| x | $f(x)$ | $g(x)$ | $(f+g)(x)$ | $f^{-1}(x)$ | $g(f(x))$ | $f(g(x))$ |
|---|---|---|---|---|---|---|
| 0 | 0 | -3 | -3 | 0 | -3 | undefined |
| 1 | 2 | -2 | 0 | undefined | -1 | undefined |
| 2 | 4 | -1 | 3 | 1 | 1 | undefined |
| 3 | 6 | 0 | 6 | undefined | 3 | 0 |
| 4 | 8 | 1 | 9 | 2 | 5 | 2 |
| 5 | 10 | 2 | 12 | undefined | undefined | 4 |
| 6 | 12 | 3 | 15 | 3 | undefined | 6 |
| 7 | 14 | 4 | 18 | undefined | undefined | 8 |
| 8 | 16 | 5 | 21 | 4 | undefined | 10 |

Question 5:

It would be good to have a whole class discussion about a reasonable domain for each of these functions based on the time it takes for each contestant to cross the finish line. The hare crosses the finish line at 10 seconds, the tortoise at approximately 6.65 seconds.

***The tortoise:*** $a(t) = 243 \cdot \left(\frac{1}{3}\right)^{\frac{1}{20} \cdot 2^t}$ ($a$ is in turtles per meter, $t$ in seconds)

$a(5) \approx 42$ turtles/meter

***The hare:*** $a(t) = 2^{\frac{1}{10}t^2}$ ($a$ is in rabbits per meter, $t$ in seconds)

$a(5) \approx 6$ rabbits/meter

***Total number of spectators d meters from the starting line*:** $a(d) = 2^{\frac{1}{10}d} + 243 \cdot \left(\frac{1}{3}\right)^{\frac{1}{20}d}$

**Aligned Ready, Set, Go: Modeling with Functions 8.6**

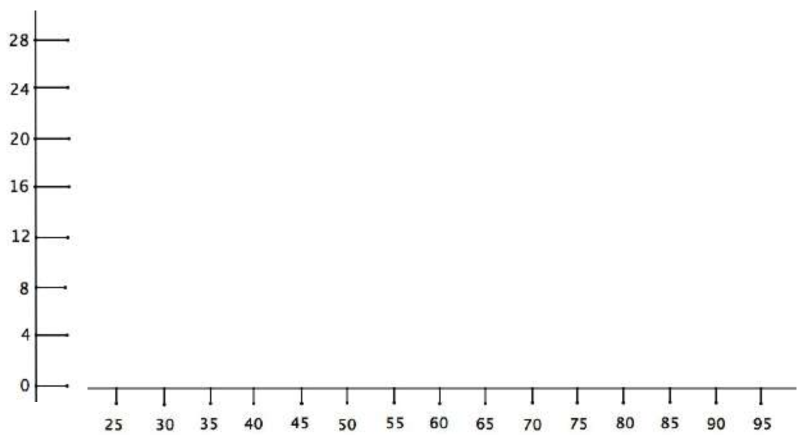
READY, SET, GO! Name Period Date

## READY

Topic: Building a histogram

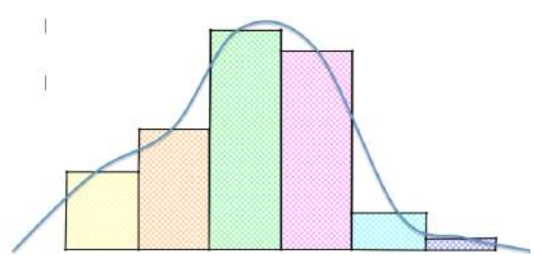
One hundred forty-four college freshmen were given a math placement exam with 100 possible points. The results show that 56 different scores were earned, ranging from 24 to 96. The scores were grouped in intervals as shown in the following table:

1. Make a histogram of the grouped data in the chart. (Note: The midpoint of each cell is given in the horizontal axis. The sides of the cells will match the score interval. Frequency is the vertical height.)




| Score interval | Midpoint of interval | Frequency of interval |
|---|---|---|
| 92.5 – 97.5 | 95 | 2 |
| 87.5 – 92.5 | 90 | 4 |
| 82.5 – 87.5 | 85 | 10 |
| 77.5 – 82.5 | 80 | 13 |
| 72.5 – 77.5 | 75 | 21 |
| 67.5 – 72.5 | 70 | 26 |
| 62.5 – 65.5 | 65 | 18 |
| 57.5 – 62.5 | 60 | 15 |
| 52.5 – 57.5 | 55 | 12 |
| 47.5 – 52.4 | 50 | 8 |
| 42.5 – 47.5 | 45 | 3 |
| 37.5 – 42.5 | 40 | 3 |
| 32.5 – 37.5 | 35 | 4 |
| 27.5 – 32.5 | 30 | 4 |
| 22.5 – 27.5 | 25 | 1 |

2. Locate the midpoint at the top of each cell in your histogram and connect each consecutive midpoint with straight line segments. The resulting figure is called a *frequency polygon.* If you smooth the line segments out into a smooth curve, you will create a *frequency curve.* Make a frequency curve on your histogram. It should look something like the figure on the right.



## SET

Topic: Identifying the 2 functions that make up a composite function

**Find functions $f$ and $g$ so that $f \circ g = H$.**

3. $H(x) = \sqrt{x^2 + 5x - 4}$

4. $H(x) = \left(3 - \frac{1}{x}\right)^2$

5. $H(x) = (3x - 7)^4$

6. $H(x) = |5x^2 - 78|$

7. $H(x) = \frac{2}{3 - x^5}$

8. $H(\theta) = (\tan\theta)^2$

9. $H(\theta) = tan(\theta^2)$

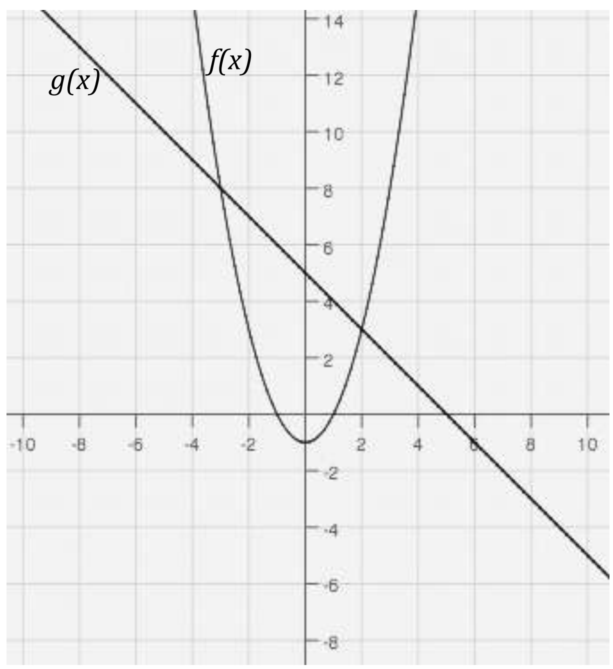
10. $H(x) = \sqrt{\frac{1}{6x}}$

11. $H(x) = 9(4x - 8) + 1$

## GO

Topic: Finding function values given the graph

**Use the graph to find all of the missing values.**




12. $f(■) = 8$

13. $g(■) = 5$

14. $f(■) = -1$

15. $g(■) = 0$

16. $f(-1) = $ ____

17. $g(0) = $ ____

18. $f(x) = g(x)$

19. $f(x) - g(x) = 0$

20. $f(x) * g(x) = 0$

21. $f(2) + g(2) = ■$

22. $f(0) - g(0) = ■$

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# 8.7H High Noon and Sunset Shadows Combined

### *A Develop Understanding Task*

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In the task *High Noon and Sunset Shadows* we described the two-dimensional circular motion of a rider on a Ferris wheel by separating that motion into two components—the horizontal motion of the "high noon shadow" of the rider as it moved along the ground, and the vertical motion of the "sunset shadow" as it moved up and down along the wall of a building. Mathematicians refer to this process as *resolving the motion into its horizontal and vertical components.*

The following data was captured by filming a person's hand as she slowly traced an image in the air with the tip of a pencil. The first table captures the horizontal movement of the pencil—similar to watching the "high noon shadow" of the pencil moving across the floor. The second table captures the vertical movement of the pencil—similar to watching the "sunset shadow" of the pencil moving up or down the wall.

| **Table 1** | |
|---|---|
| time (*sec*) | horizontal position (*inches*) |
| 0 | 2 |
| 1 | 4 |
| 2 | 6 |
| 3 | 4 |
| 4 | 6 |
| 5 | 9 |
| 6 | 11 |
| 7 | 12 |
| 8 | 10 |
| 9 | 8 |
| 10 | 11 |
| 11 | 12 |
| 12 | 10 |
| 13 | 8 |
| 14 | 4 |

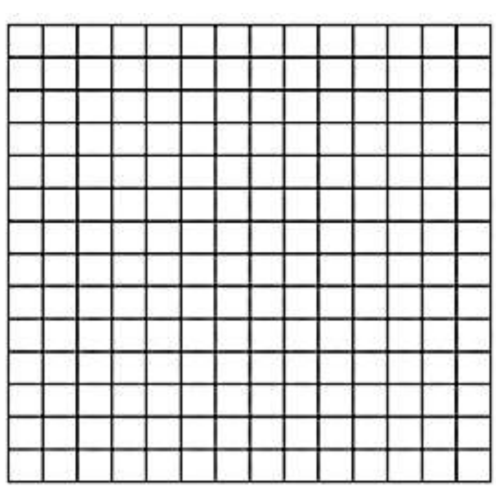
| **Table 2** | |
|---|---|
| time (*sec*) | vertical position (*inches*) |
| 0 | 11 |
| 1 | 9 |
| 2 | 12 |
| 3 | 3 |
| 4 | 12 |
| 5 | 13 |
| 6 | 12 |
| 7 | 10 |
| 8 | 7 |
| 9 | 8 |
| 10 | 7 |
| 11 | 4 |
| 12 | 2 |
| 13 | 2 |
| 14 | 3 |

1. Examine table 1 and describe what it tells you about the horizontal motion of the person's hand.

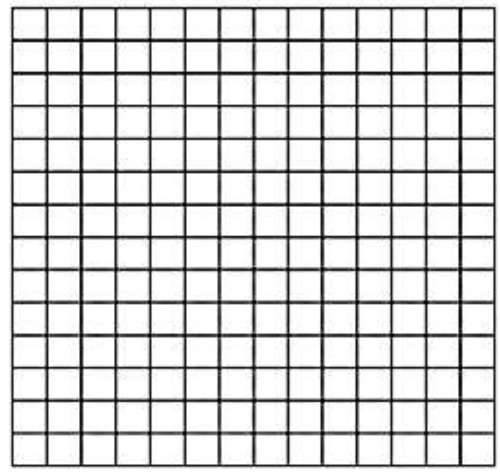
2. Examine table 2 and describe what it tells you about the vertical motion of the person's hand.

3. The person is tracing a familiar letter in the air. Can you guess what the letter is? Explain how you made your guess.

4. At each second we know the horizontal and vertical location of the tip of the pencil. Plot points on the grid below to indicate these locations. Connect these points in a way that would show the location of the pencil at instances in time between the seconds given. Based on this graph, what letter do you think the person was tracing?



5. Draw a two-dimensional figure on the grid at the right. Create tables, like the ones above, to indicate where your pencil was at different moments in time as you drew your figure. Trade your tables with a partner and see if you can each replicate the figure that the other person drew based on the data you received.



| **Table 1** | |
|---|---|
| time (*sec*) | horizontal position (*inches*) |
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | |
| 9 | |
| 10 | |
| 11 | |
| 12 | |
| 13 | |
| 14 | |

| **Table 2** | |
|---|---|
| time (*sec*) | vertical position (*inches*) |
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| 7 | |
| 8 | |
| 9 | |
| 10 | |
| 11 | |
| 12 | |
| 13 | |
| 14 | |

Let's return to the rider on the Ferris wheel. The horizontal and vertical positions of the rider as a function of time are given by the following equations:

$$x(t) = 25\cos\left(\tfrac{\pi}{10}t\right)$$
$$y(t) = 25\sin\left(\tfrac{\pi}{10}t\right) + 30$$

6. How can you use these equations to determine the location of the rider at any instant in time? For example, how might you complete the following table?

| time (seconds) | position of the rider |
|---|---|
| 0 | |
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 7.5 | |
| 10 | |
| 12.5 | |
| 15 | |
| 17.5 | |
| 20 | |

# 8.7H High Noon and Sunset Shaldows Combined – Teacher Notes

## *A Develop Understanding Task*

**Purpose**: This task introduces students to the idea of sketching curves that have been defined parametrically. That is, curves for which the horizontal and vertical components of the position have been defined by either tables or equations. Students will also relate the one-dimensional horizontal and vertical numerical data or function rules to the two-dimensional motion they describe.

**Core Standards Focus**:

**F.BF.1** Write a function that describes a relationship between two quantities.★

UT Honors: *Define a curve parametrically and draw parametric graphs.*

**Standards for Mathematical Practice:**
**SMP 4 – Model with mathematics**
**SMP 5 – Use appropriate tools strategically**
**Vocabulary:** The task continues to build understanding of *horizontal and vertical components* of two-dimensional motion.

**The Teaching Cycle:**
**Launch (Whole Class):**
Introduce students to the task by reminding them of their experience with the rider on the Ferris wheel. In those tasks we were able to describe the horizontal and vertical motion of the rider's shadow as the shadow moved horizontally along the ground when the sun was directly overhead—the "high noon shadow"—or vertically up and down a wall when the sun was on the horizon—the "sunset shadow." Point out that it is interesting that we can describe the motion of these shadows with functions, but we do not yet have a way to algebraically describe the motion of

the rider himself, since the rider is moving in two-different dimensions at any instant in time. In this task we will begin to resolve this dilemma.

Have students use a pencil to trace their signature in the air. This is fairly complex two-dimensional motion. If we observed this motion in a strobe light—or took pictures of this motion with a camera at equal intervals of time—we would see discrete positions where the pencil is located at specific instances in time. Inform students that the data in tables 1 and 2 are taken from such an experiment, but the two students recording the data focused on two different relationships. One collected data about the horizontal position of the tip of the pencil, the other collected data about the vertical position. Following this introduction, set students to work on the remainder of the task.

**Explore (Small Group):**

Listen for how students describe the horizontal and vertical motion of the hand. Do they notice that the hand moves back and forth horizontally multiple times, but up and down only a couple of times? Can they identify when the hand was moving more slowly or more rapidly in either the horizontal or vertical direction? Are there times when the hand is moving slowly in one direction, but more rapidly in the other?

Make sure that students spend sufficient time thinking about questions 1-3 and discussing these questions with their partner before they plot the data in question 4. Don't allow them to move to question 4 until they have made a prediction as to what cursive letter is being described by this set of data.

Students still might not recognize the letter after plotting the data points. It is important to imagine the motion between points and attend to the order in which the points are connected.

Once they have interpreted the data correctly, and recognize that it traces out the cursive letter B, let students move on to the partner work of plotting their own figure on the grid (question 5). Point out that the figure needs to be one they can draw without lifting their pencil. Each student should create two tables to resolve the continuous two-dimensional motion of drawing the figure into its horizontal and vertical components. Have students exchange data tables and try to recreate their partner's drawings. See if they can do so without actually plotting the points, but rather by envisioning the combined motion of the two components.

Give students time to work on the Ferris wheel problem (question 6). Watch for how students describe the position of the rider—do they list the horizontal and vertical components of the position separately (e.g., $x = 0$, $y = 55$ for $t = 5$), or do they describe the position as a single ordered-pair $(0, 55)$?

**Discuss (Whole Class):**

Select one or two student-generated tables to place on the board. See if the rest of the class can predict what the figure looks like without actually plotting points. Have them describe what the motion of the shadow along the ground might look like, based on the data, and then the motion of the shadow on the wall. Ask students to determine intervals of time where the horizontal shadow is moving slowly left or right and describe the corresponding vertical motion during those same intervals. Continue to examine small intervals of time until students can predict the actual shape of the two-dimensional graph.

End the discussion by asking students to describe different ways they could fill out the second column of the Ferris wheel table to describe the position of the rider. Make sure that students identify that a single ordered-pair works, rather than just describing the two separate components of the position.

**Aligned Ready, Set, Go: Modeling with Functions 8.7H**

READY, SET, GO! Name Period Date

## READY

Topic: Writing equations of lines

**Write the equation of the line (in slope-intercept form) that is defined by the given information.**

1. $A(5,9)B(7,17)$

2. $P(-3,8)Q(-4,13)$

3. $G(3,-10)H(1,-11)$

4. $L(-5,6)M(-8,8)$

5.

| x | y |
|---|---|
| 1 | -1 |
| 6 | 1 |
| 11 | 3 |

6.

| x | y |
|---|---|
| 1 | 2 |
| 5 | 5 |
| 9 | 8 |

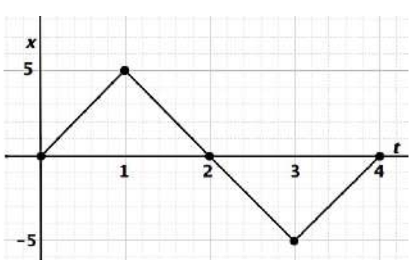
## SET

Topic: Graphing parametric equations

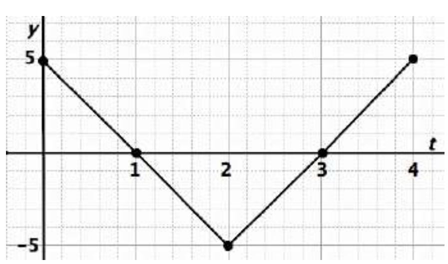
**The two given graphs show the motion of a particle whose position at time *t seconds* is given by $x = f(t)$ *and* $y = g(t)$. Describe the motion of the particle. Then graph the two graphs as one graph in the $xy$ plane. Connect the points to indicate the motion at each second.**

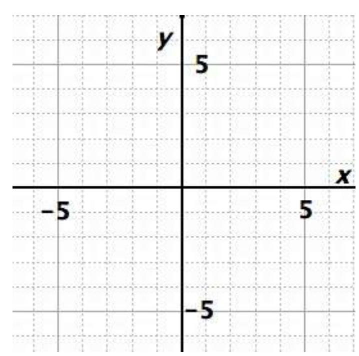
7.

*f (t)*



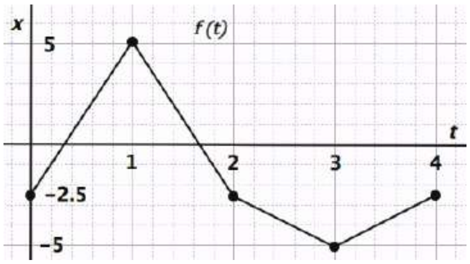

*g(t)*

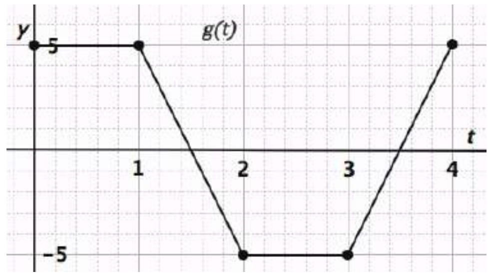


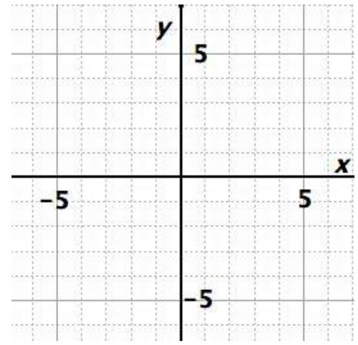




Describe the motion:

8.

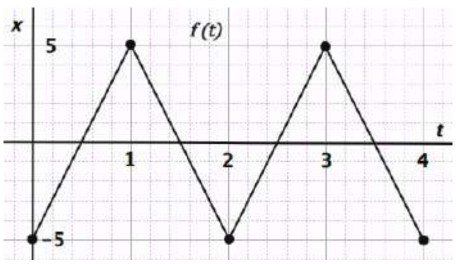


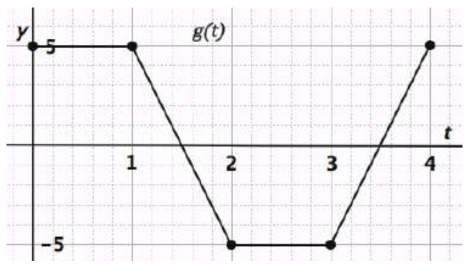



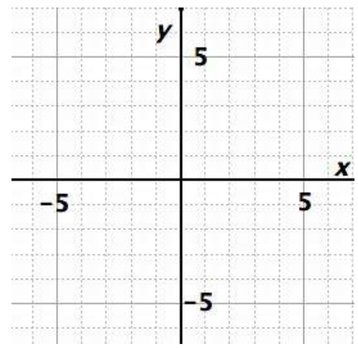




Describe the motion:

9.



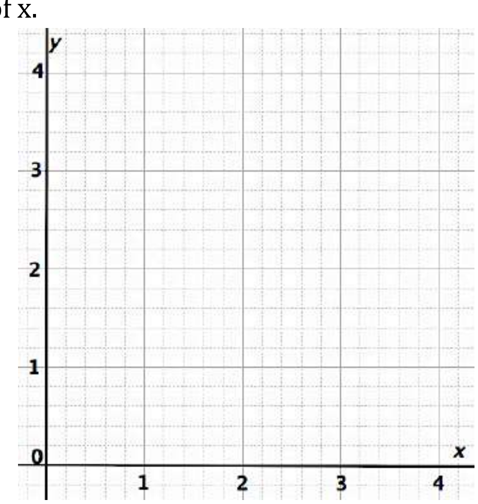




Describe the motion:

10. Fill in the table of values for the pair of parametric equations: $x = 3 + \sin t$ *and* $y = 2 + \cos t$. Sketch the graph of the two equations in the *xy*-plane. Indicate the direction of the curve. Use the graph to write an equation for y as a function of x.

| time $(t)$ | $x = 3 + \sin t$ | $y = 2 + \cos t$ | ( x, y ) |
|---|---|---|---|
| 0 | | | |
| 1 | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |
| 5 | | | |
| 6 | | | |
| 7 | | | |



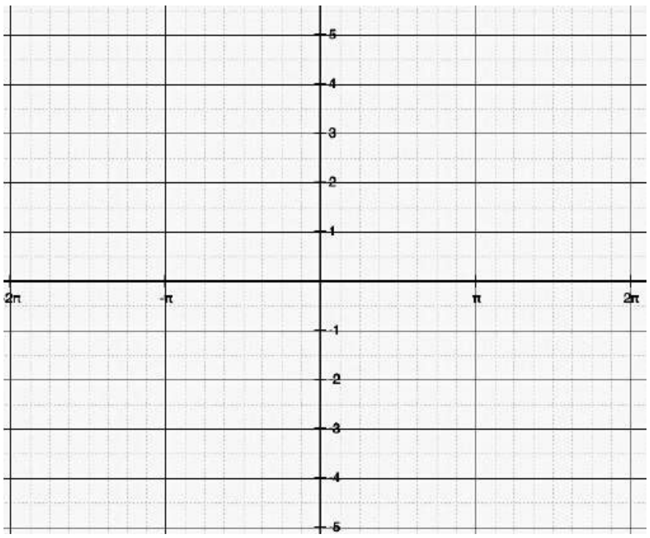

equation:

## GO

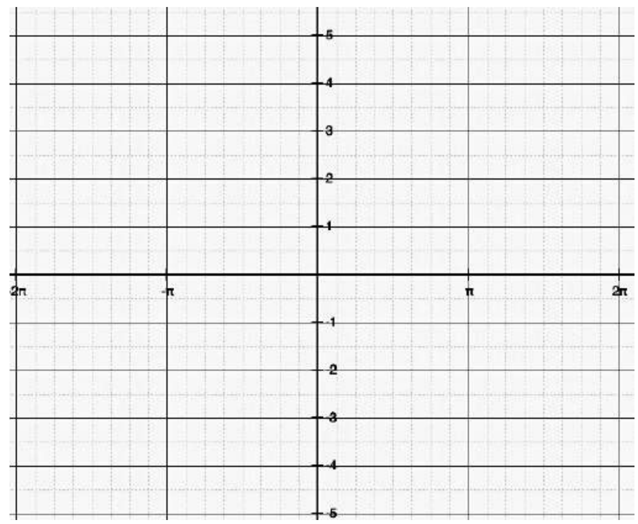
Topic: Graphing trigonometric functions

**Graph the functions.** $(-2\pi \leq \theta \leq 2\pi)$

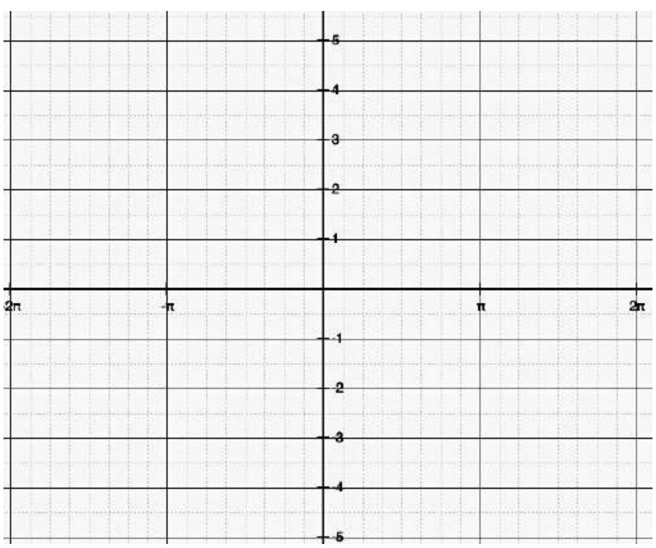
11. $y = 4 + \sin\theta$



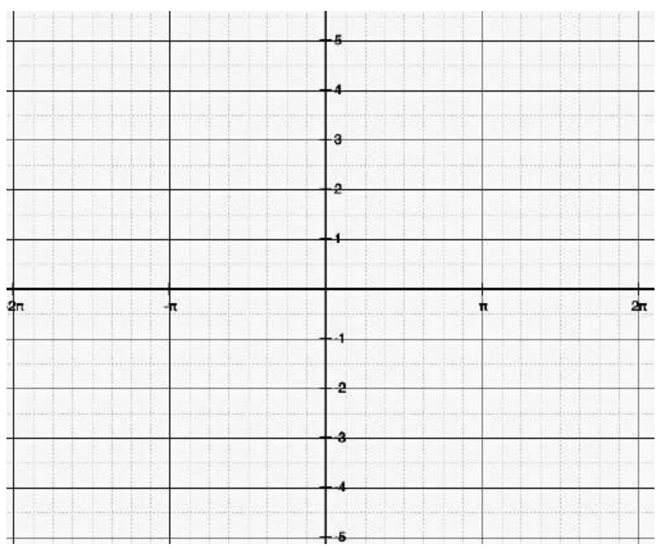
12. $y = \tan\theta$



13. $y = -3 + 2\cos\theta$

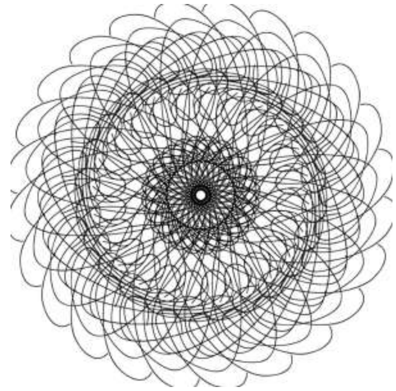


14. $y = -1 - 3\cos\theta$



# 8.8H Parametrically-Defined Curves

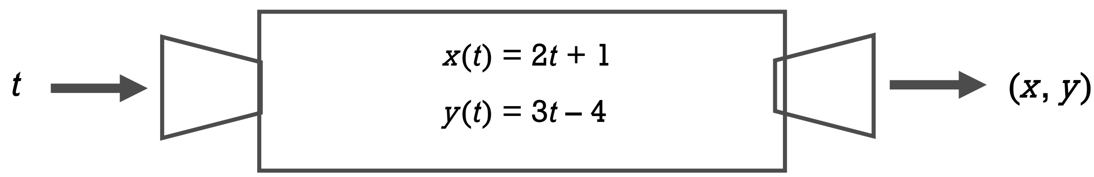
## *A Solidify and Practice Understanding Task*





A **parametric curve** is defined by a set of equations that give the coordinates of points on the curve in terms of another variable called a *parameter*.

The following diagram suggests that a parametric curve can be defined as a function in which the input is a parameter variable $t$ and the output is an ordered-pair $(x, y)$. The rule that defines the output is a set of equations defined by $x(t)$ and $y(t)$.




1. The parametric curve defined in the diagram is a line.

   a. Find the slope of the line.

   b. How does the slope of the line show up in its parametric representation?

   c. Find the $y$-intercept of the line.

   d. How does the $y$-intercept of the line relate to the parametric representation?

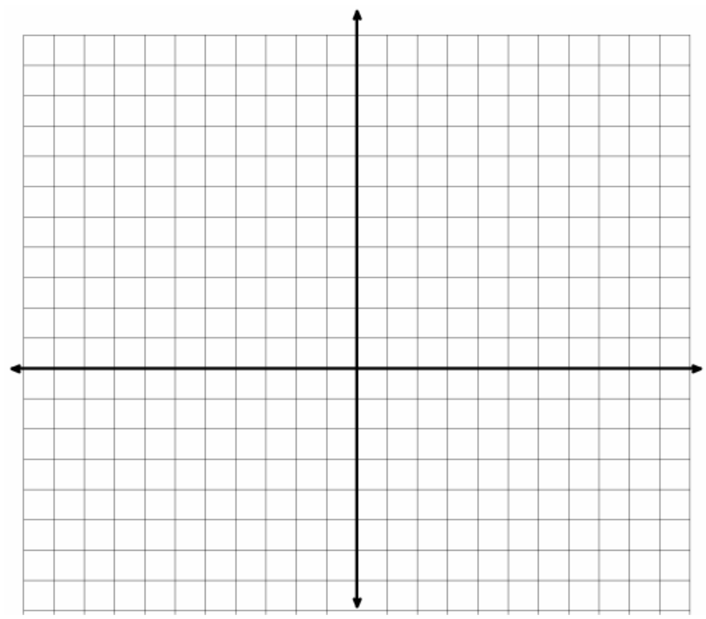
   e. Write the equation of the line $y = 2x + 3$ in parametric form.

   f. Write the equation of the line $y = ½ (x – 2) + 4$ in parametric form.

2. Given the following two parametrically-defined curves defined on the interval $-3 \le t \le 3$:

$$x_1(t) = 2t \qquad\qquad x_2(t) = 2t + 2$$
$$y_1(t) = t^2 \quad \text{and} \quad y_2(t) = t + 3$$

a. Graph the two curves on the same coordinate grid.



b. Each of these two curves is an easily recognizable function. Write the standard function equation for each of the curves.

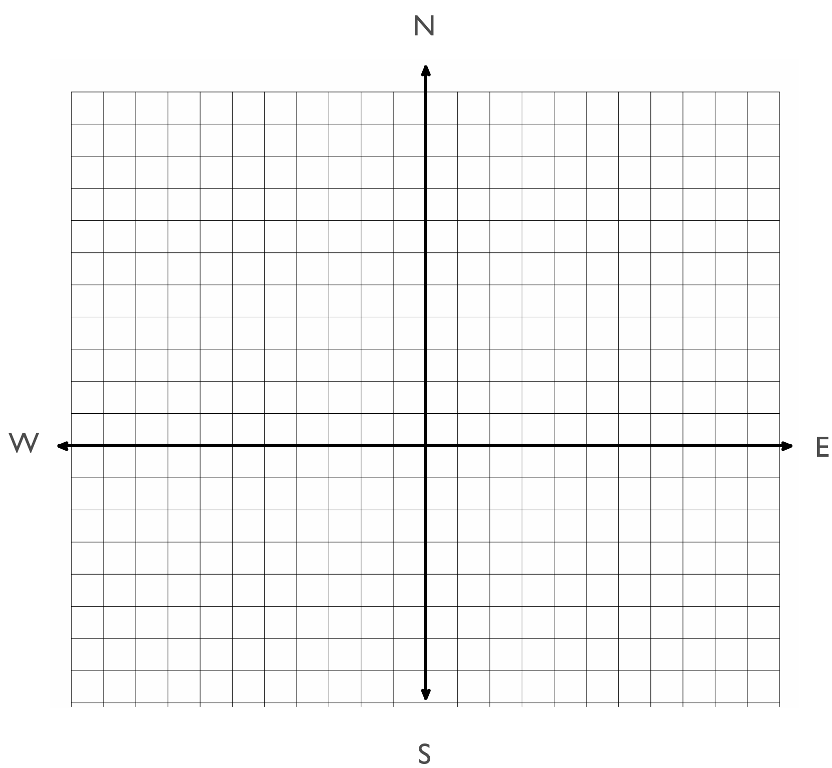
c. Examine the relationship between the parametric form for each of these curves written in terms of the parameter $t$ and its standard function form. Can you suggest a strategy for eliminating the parameter $t$ in the parametric form of each curve to obtain the more familiar function form?

d. Find the points of intersection of these two curves.

3. A flight controller is monitoring the flight paths of two planes, as given by the following two parametrically defined functions. The control tower is located at the origin and horizontal and vertical distances are measured in miles.

$$x_1(t) = 200t \qquad\qquad x_2(t) = -300t + 900$$
$$y_1(t) = -200t + 800 \quad \text{and} \quad y_2(t) = -100t + 700$$

a. Graph the projected paths for each plane if they continue on their same course for the next four hours (that is, on the interval of time $0 \le t \le 4$).




b. Should the flight controller be concerned about a possible midair collision? Explain your reasoning.

# 8.8H Parametrically-Defined Curves – Teacher Notes

## *A Solidify and Practice Understanding Task*

**Purpose**: The purpose of this task is to more formally define parametric curves and to illustrate how such curves can be thought of as relationships between an input parameter $t$ and an output that consists of an ordered-pair $(x, y)$. Consequently, parametrically-defined curves are a new kind of function where each value of $t$ is mapped onto a unique point in the plane. The graph of a parametrically-defined curve consists of the set of ordered-pairs that are generated by the pair of equations $x(t)$ and $y(t)$. However, unlike function graphs, parametric curves have an orientation and dynamic trajectory—as $t$ increases the curve is extended and its path may even cross over itself for different values of $t$. Although the parameter $t$ is often considered to represent time, it can have different interpretations. In this task, it will be helpful to consider the parameter $t$ as representing time, and treating each curve as being traced out over an interval of time as $t$ increases. This will help students recognize that although the paths of the curves appear to intersect, the apparent points of intersection do not necessarily occur at the same instances in time. This becomes most apparent in question 3.

**Core Standards Focus**:

**F.BF.1** Write a function that describes a relationship between two quantities.★

**MVP Honors**: *Define a curve parametrically and draw parametric graphs.*

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**Standards for Mathematical Practice:**

**SMP 4 – Model with mathematics**

**Vocabulary:** This task defines *parametric curves* as functions that maps the quantity $t$, representing time, to and ordered-pair $(x, y)$, representing a location on a plane.

**The Teaching Cycle:**

**Launch (Whole Class):**

Remind students of their work on question 6 in the previous task where they created a table of values for the relationship between elapsed time $t$ and the location of the rider on the Ferris wheel. Point out that this relationship is a function since at each instant in time the rider is at a unique position. (Although the rider returns to the same position for each revolution of the Ferris wheel, the elapsed time is different. This is similar to function graphs in which two different input values of $x$ may produce the same output value $y$.) Relate this function idea to the diagram given at the beginning of this task. Then set students to work on questions 1-3.

**Explore (Small Group):**

Since there is no interval given for $t$, students may struggle with how to begin thinking about the curve represented by the parametric equations given in the diagram. Suggest they might do what they have always done when they encounter a new function: plot a few points to get a feel for the graph. In this case, they will need to select input values for $t$, perhaps starting at $t = 0$, or $t = -2$, or with whatever value they would like, and then examining other values of $t$ in some meaningful organized way to see what happens. Suggest also that they plot the ordered pairs produced by each choice of different values of $t$. This experimentation with plotting parametric curves is an important part of developing an understanding of their construction.

For question 1, listen to how students relate the slope of the graph of the line, $^3\!/_2$, to the coefficients of $x$ and $y$ in the parametric equations. Do they recognize that for each unit of increase in the parameter $t$ the $x$-coordinate will increase by 2 and the $y$-coordinate will increase by 3, as determined by the coefficients? If students do not make this observation in question 1a—perhaps only determining the slope of the line from its graph—suggest that they examine a table of values laid out in four columns: a $t$ column, an $x$ column, a $y$ column, and a column representing the ordered-pairs $(x, y)$. The first and last columns of this table represent the inputs and outputs of this parametrically-defined function. The second and third columns represent the more familiar input and output values for a linear function in standard form.

| *t* | *x* | *y* | **(*x*, *y*)** |
|---|---|---|---|
| -2 | -3 | -10 | (-3, -10) |
| -1 | -1 | -7 | (-1, -7) |
| 0 | 1 | -4 | (1, -4) |
| 1 | 3 | -1 | (3, -1) |
| 2 | 5 | 2 | (5, 2) |
| 3 | 7 | 5 | (7, 5) |

For question 1c students need to recall that the $y$-intercept of the line occurs when $x = 0$. By inspection, this happens when $t = -½$. Students will need to find the corresponding $y$ value.

The lines in questions 1e and 1f do not have unique parametric forms. Students might use guess and check or more sophisticated strategies based on what they have observed about the parametric form of the line examined in question 1a-1d. You will want to make note of the different strategies that emerge, and the different forms students produce for their parametric curves.

Depending on your students, you may want to hold a whole class discussion of the issues raised in question 1 before moving onto questions 2 and 3.

Question 2 examines two important ideas about parametric curves: finding a strategy for eliminating the parameter $t$ so that the curve can be written in the form $y = f(x)$, and examining the dynamic trajectory of points moving along parametric curves, including what it means for two parametric curves to intersect. After students have answered question 2b, you may need to prompt their work on question 2c by asking, "Is there a way we can eliminate the parameter $t$ so we can relate the variables $y$ and $x$ to each other directly, similar to the equations you wrote in question 2b?"

Students will probably look at their graphs to answer question 2d and note that the points (-2, 1) and (4, 4) appear on both paths. Some students may be concerned that these points are generated by different values of $t$. Allow students to debate this issue, if it arises, but you do not need to press for this issue to surface at this time. It will arise naturally in question 3, and then you can have students go back and grapple with this idea in question 2.

**Discuss (Whole Class):**
Focus the first part of the discussion on linear functions and how they can be written in parametric form by having some students discuss how they determined their parametric equations for questions 1e and 1f. A trivial way of writing the parametric form of a line would consist of the following equations: $x(t) = t$ and $y(t) = mt + b$. If a student has used this method have that student share first. (If not, you should choose to share this method at some point in the discussion.) Point out that this strategy allows us to write any of our familiar families of functions in parametric form. Then move to a strategy that attends to describing horizontal and vertical components of motion for $x(t)$ and $y(t)$. In general, students should be able to generalize that a parametric form of a line looks like $x(t) = at + b$ and $y(t) = ct + d$ where the slope of the line is given by $m = {}^{c}/_{a}$ and the point $(b, d)$ is on the line.

Shift the focus of the discussion to the strategy for eliminating the variable $t$ from the parametric equations so that the function can be written in the form $y = f(x)$. Allow students to share their strategies for answering question 2c, leading to an efficient strategy such as solving for the parameter $t$ in terms of $x$ and then substituting this expression into the $y$-equation for $t$.

Move to question 3 and allow students to make an argument that even though the linear functions representing the paths of the two airplanes intersect, the airplanes do not pass through that point at the same instant in time. Ask the question, "So, should we say that the point (300, 500) is a point of intersection of the two parametric curves?" It may help to resolve this question by examining the follow table:

| **Elapsed time, *t hours*** | **Location of plane 1** | **Location of plane 2** |
|---|---|---|
| 0 | (0, 800) | (900, 700) |
| 0.5 | (100, 700) | (750, 650) |
| 1 | (200, 600) | (600, 600) |
| 1.5 | (300, 500) | (450, 550) |
| 2 | (400, 400) | (300, 500) |
| 2.5 | (500, 300) | (150, 450) |
| 3 | (600, 200) | (0, 400) |
| 3.5 | (700, 100) | (-150, 350) |
| 4 | (800, 0) | (-300, 300) |

Students should note that there is no input value *t* for which the outputs of the two parametric functions are the same.

**Aligned Ready, Set, Go: Modeling with Functions 8.8H**

READY, SET, GO! Name Period Date

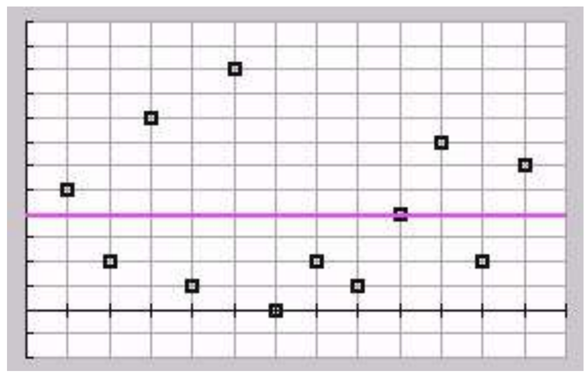
## READY

Topic: Recalling measures of central tendency

1. Find the mean, median, and mode of the following test scores:

   98, 74, 70, 68, 85, 82, 85, 94, 90, 91, 99, 85, 88, 79, 96, 98, 85, 82, 80, 86

2. The graph to the right shows 12 points whose **y-values** add up to 48. The y = 4 line is graphed. <u>Use the position of the points on the graph</u> to explain why 4 is the average of the 12 y - values.



## SET

Topic: Making sense of parametric equations

**For each set of parametric equations,**
**a) Create a table of x- and y-values**
**b) Plot the points (x, y) in your table and sketch the graph.** (Indicate the direction of the curve.)
**c) Find the rectangular equation.**

3. $x = 3t - 3$
   $y = 2t + 1$

   Rectangular equation:

| $t$ | $x$ | $y$ |
|---|---|---|
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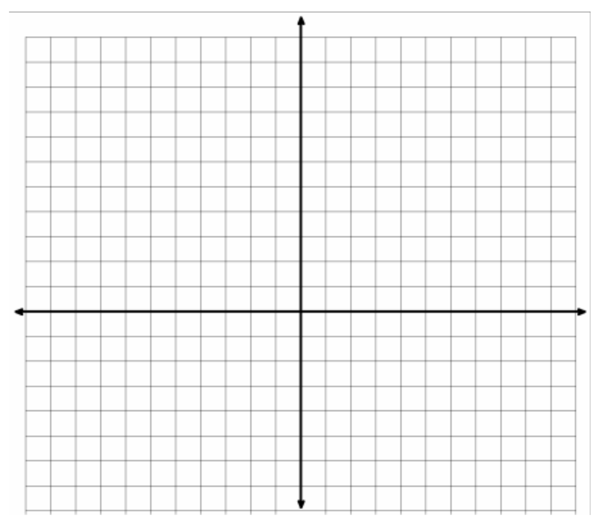

4. $x = t + 2$
$y = t^2$

Rectangular equation:

| $t$ | x | y |
|---|---|---|
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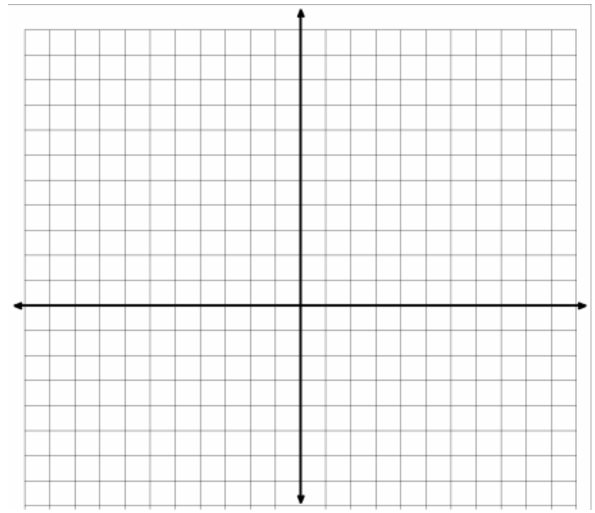

5. $x = 5 + 2\cos t$
$y = 3 + 2\sin t$

Rectangular equation:

| $t$ | x | y |
|---|---|---|
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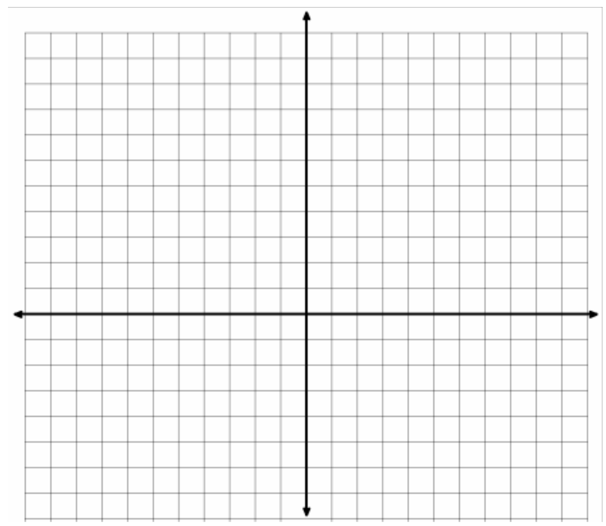

**Match the parametric equations with their graphs.**
(You may use a calculator.)

6. $x = 5\cos t$ *and* $y = \sin 3t$

7. $x = 6\sin(4t)$ *and* $y = 4\sin(6t)$

8. $x = 4\,cos^3 t$ *and* $y = 4\,sin^3 t$

9. $x = \frac{1}{2}(\cos t + t\sin t)$ *and* $y = \frac{1}{2}(\sin t - t\cos t)$

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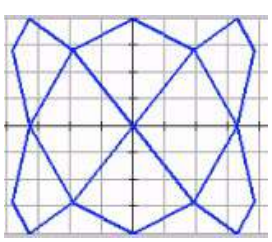

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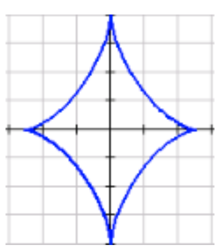

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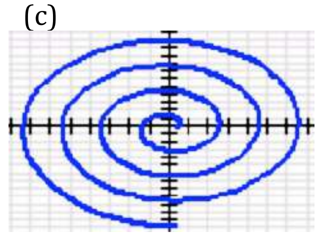

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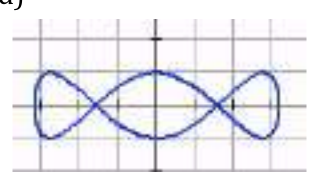

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## GO

Topic: Reviewing composition of functions

10. Given $f(x) = 3x + 2$ $and$ $g(x) = 4 - x^2$, find the following.

a) $(f \circ g)(x)$ b) $(g \circ f)(x)$ c) $(f \circ g)(-4)$ d) $(g \circ f)(-2)$

11. Is there a restriction on the domain in any of the exercises (a – d) in problem 10? Explain.

12. Given $f(x) = \frac{3x+2}{x-1}$ $and$ $g(x) = 3x + 3$, find the following.

a) $(f \circ g)(x)$ b) $(g \circ f)(x)$ c) $(f \circ g)(10)$ d) $(g \circ f)(4)$

13. Is there a restriction on the domain in any of the exercises (a – d) in problem 12? Explain.