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Transforming Mathematics Education

SECONDARY MATH THREE

An Integrated Approach

MODULE 3

Polynomial Functions

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The Mathematics Vision Project
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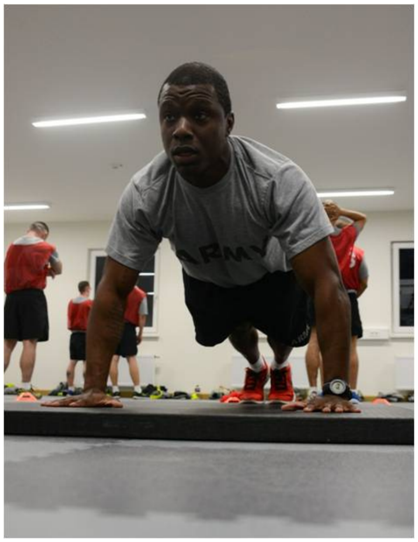
3.1 Scott's March Madness

A Develop Understanding Task



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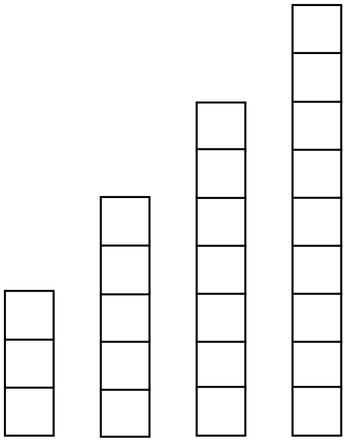
Each year, Scott participates in the "Macho March" promotion. The goal of "Macho March" is to raise money for charity by finding sponsors to donate based on the number of push-ups completed within the month. Last year, Scott was proud of the money he raised, but was also determined to increase the number of push-ups he would complete this year.

Part I: Revisiting the Past

Below is the bar graph and table Scott used last year to keep track of the number of push-ups he completed each day, showing he completed three push-ups on day one and five push-ups (for a combined **total** of eight push-ups) on day two. Scott continued this pattern throughout the month.

n Days	$f(n)$ Push-ups each **day**	$g(n)$ **Total** number of pushups in the month
1	3	3
2	5	8
3	7	15
4	9	24
5	11	35
…	…	
n		



1. Write the recursive and explicit equations for the number of push-ups Scott completed **on any given day** last year. Explain how your equations connect to the bar graph and the table above.

2. Write the recursive and explicit equation for the **accumulated total number of push-ups Scott completed by any given day** during the “Macho March” promotion last year.

Part II: March Madness

This year, Scott’s plan is to look at the total number of push-ups he *completed for the month* last year ($g(n)$) and do that many push-ups each day ($m(n)$).

n Days	$f(n)$ Push-ups each **day last year**	$g(n)$ **Total** number of pushups in the month	$m(n)$ Push-ups each **day this year**	$T(n)$ **Total** push-ups completed for the month
1	3	3	3	
2	5	8	8	
3	7	15	15	
4	9	24		
5				
…	…			
n				

3. How many push-ups will Scott complete on day four? How did you come up with this number? Write the recursive equation to represent the total number of push-ups Scott will complete for the month on any given day.

4. How many **total** push-ups will Scott complete for the month on day four?

5. Without finding the explicit equation, make a conjecture as to the type of function that would represent the explicit equation for the total number of push-ups Scott would complete on any given day for this year's promotion.

6. How does the rate of change for this explicit equation compare to the rates of change for the explicit equations in questions 1 and 2?

7. Test your conjecture from question 5 and justify that it will always be true (see if you can move to a generalization for all polynomial functions).

READY, SET, GO! Name Period Date

READY

Topic: Completing inequality statements

For each problem, place the appropriate inequality symbol between the two expressions to make the statement true.

$If\ a > b, then:$

1. $3a$ ___ $3b$
2. $b - a$ ___ $a - b$
3. $a + x$ ___ $b + x$

$If\ x > 10, then:$

4. x^2 ___ 2^x
5. $\sqrt{x}$ ___ x^2
6. x^2 ___ x^3

$If\ 0 < x < 1$

7. x ___ x^2
8. $\sqrt{x}$ ___ x
9. x ___ $3x$

SET

Topic: Classifying functions

Identify the type of function for each problem. Explain how you know.

10

x	$f(x)$
1	3
2	6
3	12
4	24
5	48

11.

x	$f(x)$
1	3
2	6
3	9
4	12
5	15

12.

x	$f(x)$
1	3
2	9
3	18
4	30
5	45

13.

x	$f(x)$
1	7
2	9
3	13
4	21
5	37

14.

x	$f(x)$
1	-26
2	-19
3	0
4	37
5	98

15.

x	$f(x)$
1	-4
2	3
3	18
4	41
5	72

16. Which of the above functions are NOT polynomials?

GO

Topic: Recalling long division and the meaning of a factor

Find the quotient without using a calculator. If you have a remainder, write the remainder as a whole number. Example: $\begin{array}{r} 7 \\ 21\overline{)149} \end{array}$ **remainder 2**

17. $30\overline{)510}$

18. $13\overline{)8359}$

19. Is 30 a factor of 510? How do you know?

20. Is 13 a factor of 8359? How do you know?

21. $22\overline{)14857}$

22. $952\overline{)40936}$

23. Is 22 a factor of 14587? How do you know?

24. Is 952 a factor of 40936? How do you know?

25. $92\overline{)3405}$

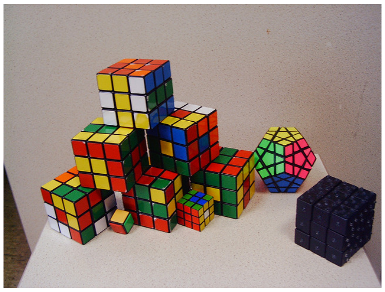
26. $27\overline{)3564}$

27. Is 92 a factor of 3405?

28. Is 27 a factor of 3564?

3.2 You-mix Cubes

A Solidify Understanding Task



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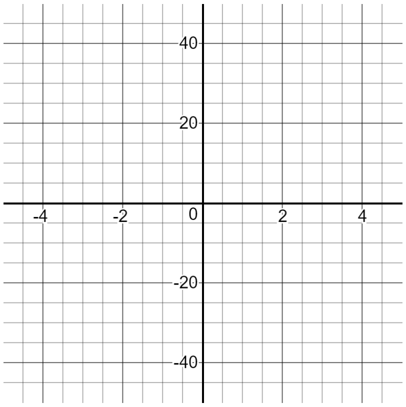
In Scott's March Madness, the function that was generated by the sum of terms in a quadratic function was called a **cubic function**. Linear functions, quadratic functions, and cubic functions are all in the family of functions called **polynomials**, which include functions of higher powers too. In this task, we will explore more about cubic functions to help us to see some of the similarities and differences between cubic functions and quadratic functions.

To begin, let's take a look at the most basic cubic function, $f(x) = x^3$. It is technically a **degree 3 polynomial** because the highest exponent is 3, but it's called a cubic function because these functions are often used to model volume. This is like quadratic functions which are **degree 2** polynomials but are called quadratic after the Latin word for square. Scott's March Madness showed that linear functions have a constant rate of change, quadratic functions have a linear rate of change, and cubic functions have a quadratic rate of change.

1. Use a table to verify that $f(x) = x^3$ has a quadratic rate of change.

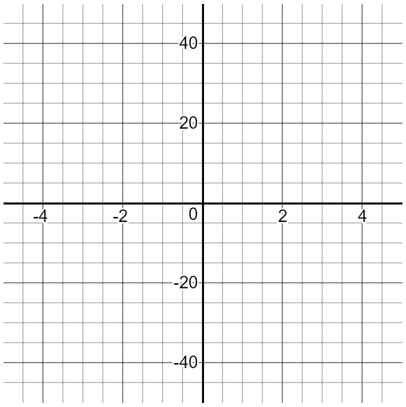
2. Graph $f(x) = x^3$.



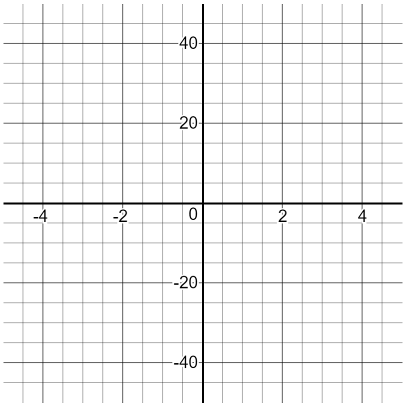
3. Describe the features of $f(x) = x^3$ including intercepts, intervals of increase or decrease, domain, range, etc.

4. Using your knowledge of transformations, graph each of the following without using technology.

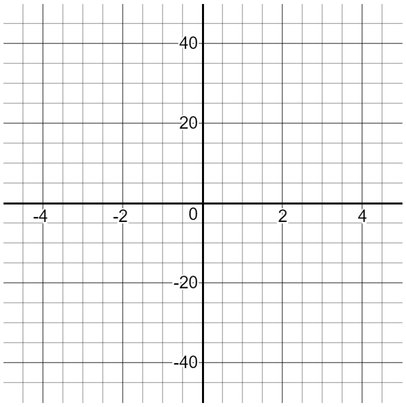
a) $f(x) = x^3 - 3$



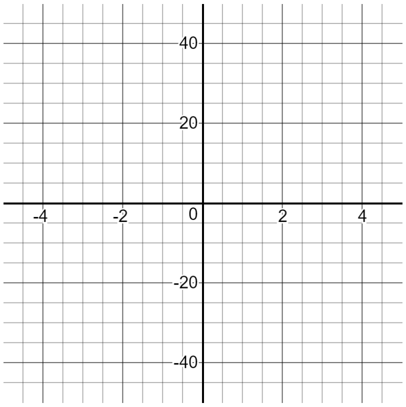
b) $f(x) = (x+3)^3$



c) $f(x) = 2x^3$



d) $f(x) = -(x-1)^3 + 2$



5. Use technology to check your graphs above. What transformations did you get right? What areas do you need to improve on so that your cubic graphs are perfect?

6. Since quadratic functions and cubic functions are both in the polynomial family of functions, we would expect them to share some common characteristics. List all the similarities between $f(x) = x^3$ and $g(x) = x^2$.

7. As you can see from the graph of $f(x) = x^3$, there are also some real differences in cubic functions and quadratic functions. Each of the following statements describe one of those differences. Explain <u>why</u> each statement is true by completing the sentence.

a) The range of $f(x) = x^3$ is $(-\infty, \infty)$, but the range of $g(x) = x^2$ is $[0, \infty)$ because:

____

b) For $x > 1$, $f(x) > g(x)$ because: ______________________________________

____

c) For $0 < x < 1$, $g(x) > f(x)$ because: ___________________________________

____

READY, SET, GO! Name Period Date

READY

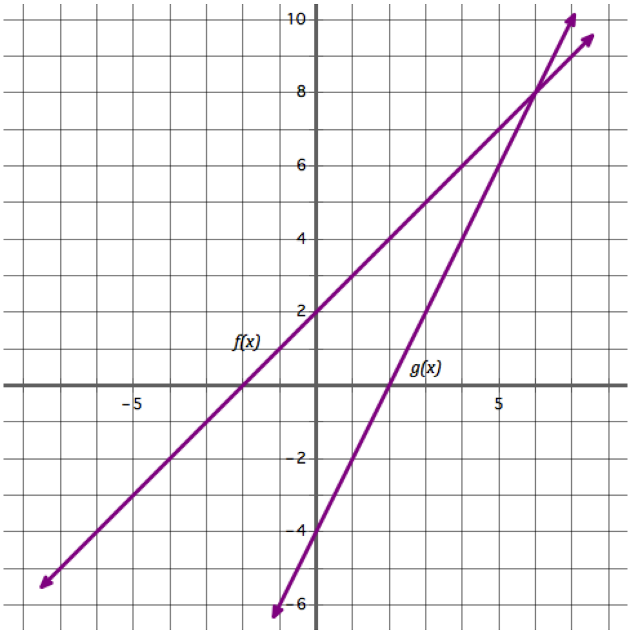
Topic: Adding and subtracting binomials

Add or subtract as indicated.

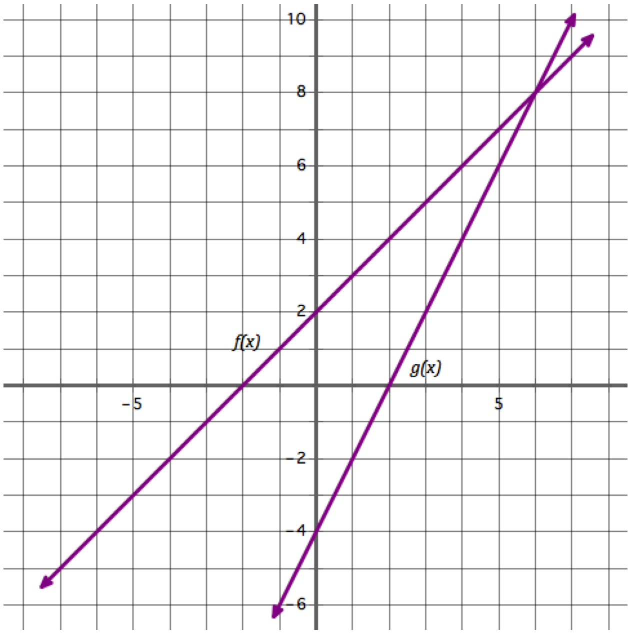
1. $(6x+3)+(4x+5)$
2. $(x+17)+(9x-13)$
3. $(7x-8)+(-2x+9)$
4. $(4x+9)-(x+2)$
5. $(-3x-1)-(2x+5)$
6. $(8x+3)-(-10x-9)$
7. $(3x-7)+(-3x-7)$
8. $(-5x+8)-(-5x+7)$
9. $(8x+9)-(7x+9)$

10. Use the graphs of $f(x)$ and $g(x)$ to sketch the graphs of $f(x)+g(x)$ and $f(x) - g(x)$.

$f(x)+g(x)$



$f(x) - g(x)$.



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SET

Topic: Comparing simple polynomials

11. Complete the tables below for $y = x$ *and* $y = x^3$ *and* $y = x^5$

x	$y = x$
-1	
0	
1	

x	$y = x^3$
-1	
0	
1	

x	$y = x^5$
-1	
0	
1	

12. What assumption might you be tempted to make about the graphs of $y = x$, $y = x^3$ *and* $y = x^5$ based on the values you found in the 3 tables above?

13. What do you really know about the graphs of $y = x$ *and* $y = x^3$ *and* $y = x^5$ despite the values you found in the 3 tables above?

14. Complete the tables with the additional values.

x	$y = x$
-1	
$-1/2$	
0	
$1/2$	
1	

x	$y = x^3$
-1	
$-1/2$	
0	
$1/2$	
1	

x	$y = x^5$
-1	
$-1/2$	
0	
$1/2$	
1	

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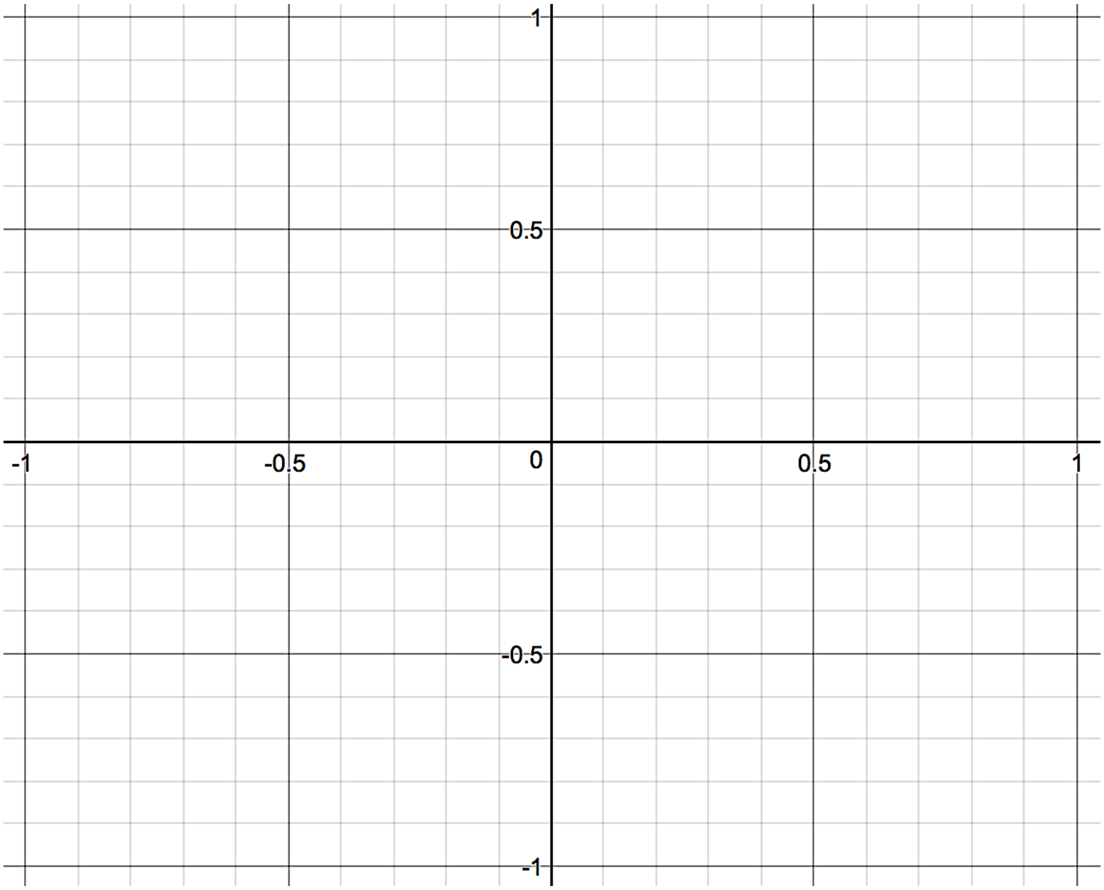
15. Graph $y = x\ and\ y = x^3\ and\ y = x^5$ on the interval $[-1, 1]$, using the same set of axes.



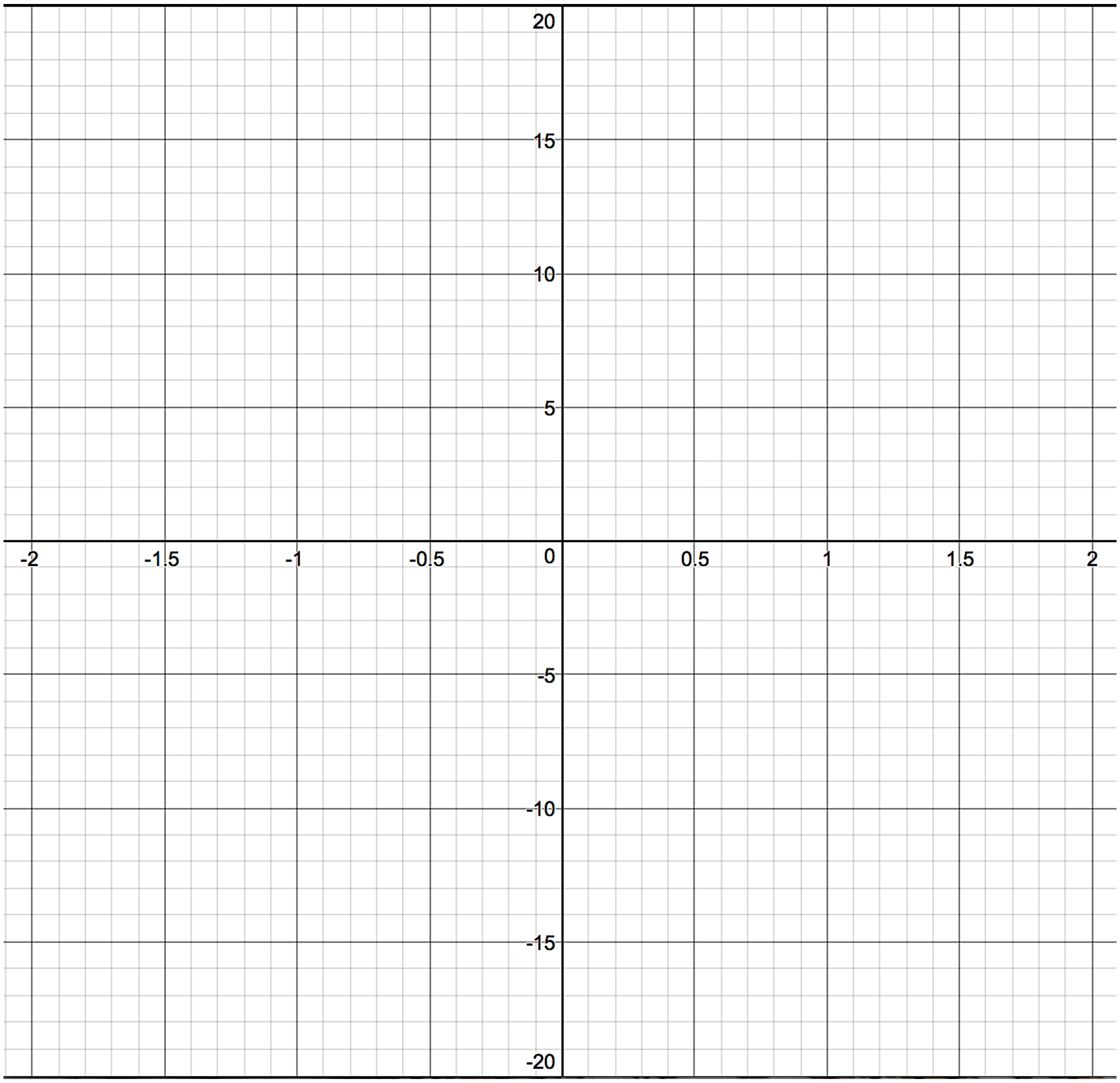
16. Complete the tables with the additional values.

x	$y = x$
-2	
-1	
$-1/2$	
0	
$1/2$	
1	
2	

x	$y = x^3$
-2	
-1	
$-1/2$	
0	
$1/2$	
1	
2	

x	$y = x^5$
-2	
-1	
$-1/2$	
0	
$1/2$	
1	
2	

17. Graph $y = x\ and\ y = x^3\ and\ y = x^5$ on the interval [-2, 2], using the same set of axes.



GO

Topic: Using the exponent rules to simplify expressions

Simplify.

18. $x^{1/3} \cdot x^{1/6} \cdot x^{1/4}$

19. $a^{2/5} \cdot a^{3/10} \cdot a^{2/15}$

20. $m^{4/7} \cdot m^{3/14} \cdot m^{5/28}$

3.3 It All Adds Up

A Develop Understanding Task

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Whenever we're thinking about algebra and working with variables, it is useful to consider how it relates to the number system and operations on numbers. Right now, polynomials are on our minds, so let's see if we can make some useful comparisons between whole numbers and polynomials.

Let's start by looking at the structure of numbers and polynomials. Consider the number 132. The way we write numbers is really a shortcut because:

$$132 = 100 + 30 + 2$$

1. Compare 132 to the polynomial $x^2 + 3x + 2$. How are they alike? How are they different?

2. Write a polynomial that is analogous to the number 2,675.

When two numbers are to be added together, many people use a procedure like this:

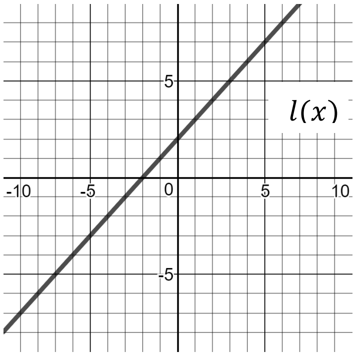
$$\begin{array}{r} 132 \\ +\ \underline{451} \\ 583 \end{array}$$

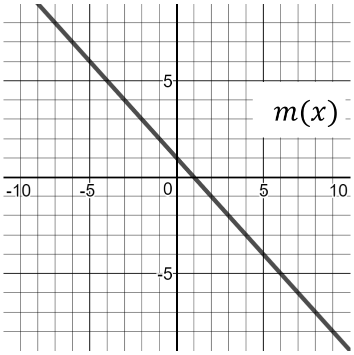
3. Write an analogous addition problem for polynomials and find the sum of the two polynomials.

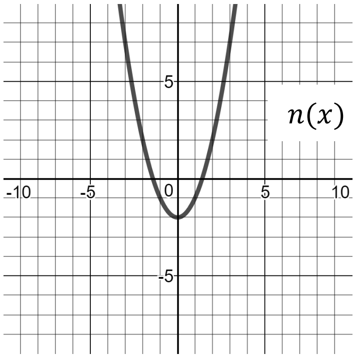
4. How does adding polynomials compare to adding whole numbers?

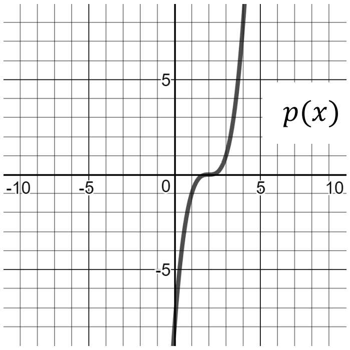
5. Use the polynomials below to find the specified sums in a-f.

$f(x) = x^3 + 3x^2 - 2x + 10$ $\quad g(x) = 2x - 1$ $\quad h(x) = 2x^2 + 5x - 12$ $\quad k(x) = -x^2 - 3x + 4$









a) $h(x) + k(x)$

b) $g(x) + f(x)$

c) $f(x) + k(x)$

d) $l(x) + m(x)$

e) $m(x) + n(x)$

f) $l(x) + p(x)$

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6. What patterns do you see when polynomials are added?

Subtraction of whole numbers works similarly to addition. Some people line up subtraction vertically and subtract the bottom number from the top, like this:

$$\begin{array}{r} 368 \\ \underline{-157} \\ 211 \end{array}$$

7. Write the analogous polynomials and subtract them.

8. Is your answer to #7 analogous to the whole number answer? If not, why not?

9. Subtracting polynomials can easily lead to errors if you don't carefully keep track of your positive and negative signs. One way that people avoid this problem is to simply change all the signs of the polynomial being subtracted and then add the two polynomials together. There are two common ways of writing this:

$$(x^3 + x^2 - 3x - 5) - (2x^3 - x^2 + 6x + 8)$$

Step 1: $= (x^3 + x^2 - 3x - 5) + (-2x^3 + x^2 - 6x - 8)$

Step 2: $= (-x^3 + 2x^2 - 9x - 13)$

Or, you can line up the polynomials vertically so that Step 1 looks like this:

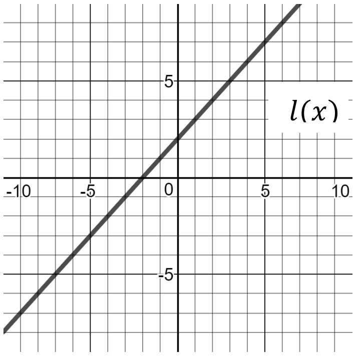
Step 1:
$$\begin{array}{r} x^3 + x^2 - 3x - 5 \\ \underline{+(-2x^3 + x^2 - 6x - 8)} \end{array}$$

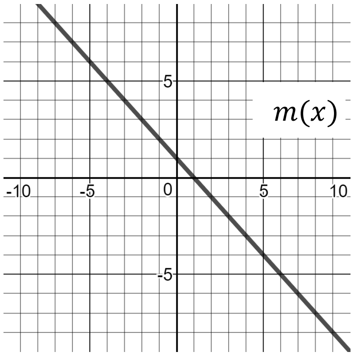
Step 2:
$$-x^3 + 2x^2 - 9x - 13$$

The question for you is: Is it correct to change all the signs and add when subtracting? What mathematical property or relationship can justify this action?

10. Use the given polynomials to find the specified differences in a-d.

$f(x) = x^3 + 2x^2 - 7x - 8$ $\quad g(x) = -4x - 7$ $\quad h(x) = 4x^2 - x - 15$ $\quad k(x) = -x^2 + 7x + 4$





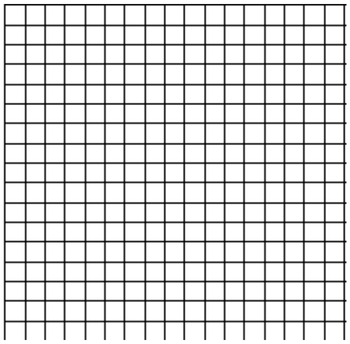
a) $h(x) - k(x)$

b) $f(x) - h(x)$

c) $f(x) - g(x)$

d) $k(x) - f(x)$

e) $l(x) - m(x)$



11. List three important things to remember when subtracting polynomials.

READY, SET, GO! Name Period Date

READY

Topic: Using the distributive property

Multiply.

1. $2x(5x^2 + 7)$
2. $9x(-x^2 - 3)$
3. $5x^2(x^4 + 6x^3)$
4. $-x(x^2 - x + 1)$
5. $-3x^3(-2x^2 + x - 1)$
6. $-1(x^2 - 4x + 8)$

SET

Topic: Adding and subtracting polynomials

Add. Write your answers in descending order of the exponents. (Standard form)

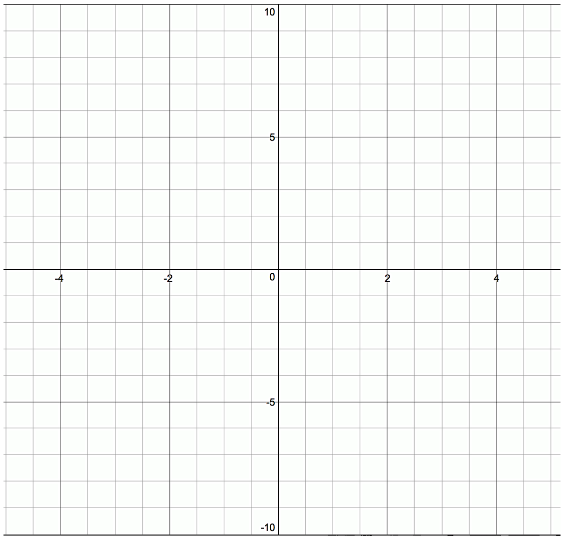
7. $(3x^4 + 5x^2 - 1) + (2x^3 + x)$
8. $(4x^2 + 7x - 4) + (x^2 - 7x + 14)$
9. $(2x^3 + 6x^2 - 5x) + (x^5 + 3x^2 + 8x + 4)$
10. $(-6x^5 - 2x + 13) + (4x^5 + 3x^2 + x - 9)$

Subtract. Write your answers in descending order of the exponents. (Standard form)

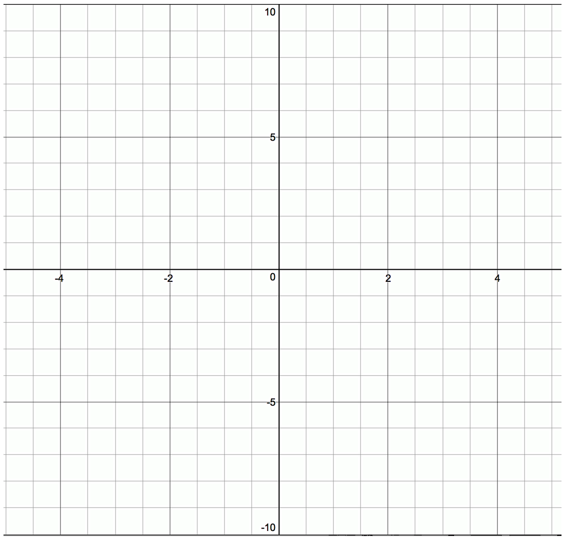
11. $(5x^2 + 7x + 2) - (3x^2 + 6x + 2)$
12. $(10x^4 + 2x^2 + 1) - (3x^4 + 3x + 11)$
13. $(7x^3 - 3x + 7) - (4x^2 - 3x - 11)$
14. $(x^4 - 1) - (x^4 + 1)$

Graph.

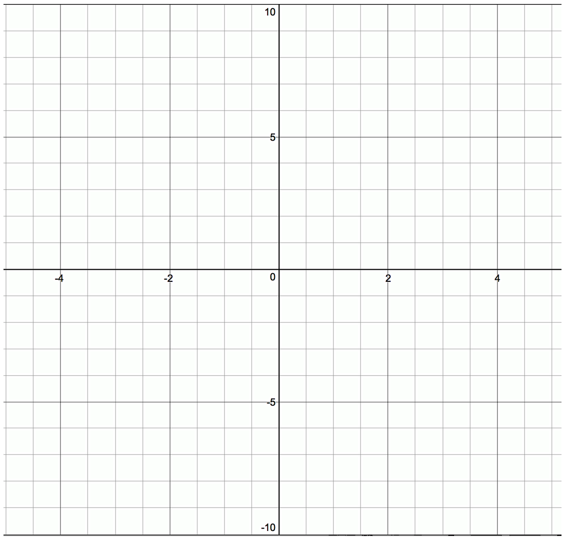
15. $y = x^3 - 2$



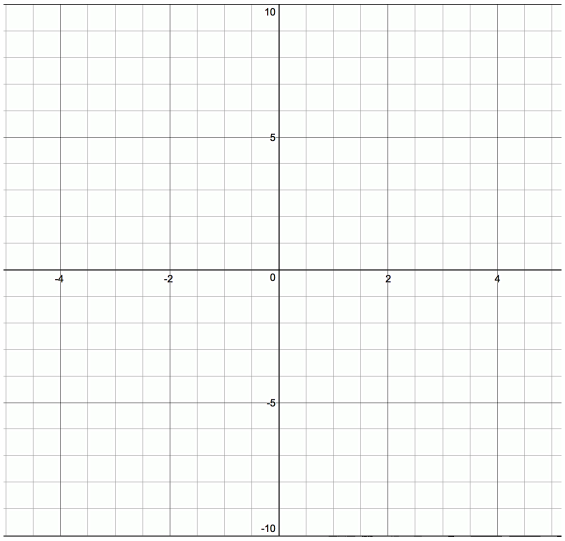
16. $y = x^3 + 1$



17. $y = (x-3)^3$



18. $y = (x+1)^3$



GO

Topic: Using exponent rules to combine expressions

Simplify.

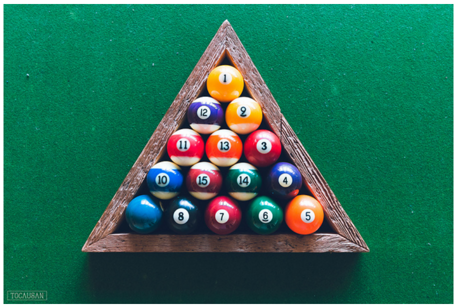
19. $x^{7/8} \cdot x^{1/4} \cdot x^{-1/2}$

20. $x^{3/16} \cdot x^{-7/8} \cdot x^{3/4}$

21. $x^{4/7} \cdot x^{2/9} \cdot x^{-1/3}$

3.4 Pascal's Pride

A Solidify Understanding Task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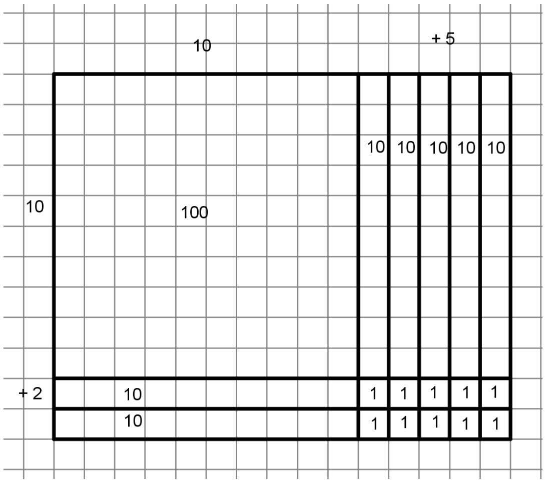
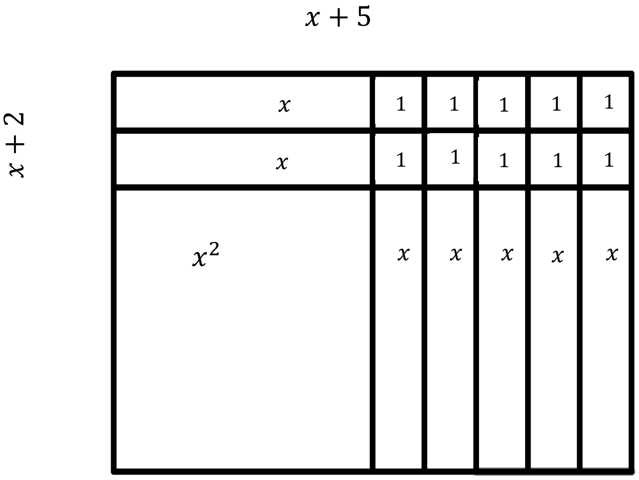
Multiplying polynomials can require a bit of skill in the algebra department, but since polynomials are structured like numbers, multiplication works very similarly. When you learned to multiply numbers, you may have learned to use an area model. To multiply 12 × 15 the area model and the related procedure probably looked like this:



$$\begin{array}{r} 12 \\ \times\ \underline{15} \\ 10 \\ 50 \\ +\ 20 \\ \underline{100} \\ \mathbf{180} \end{array}$$

You may have used this same idea with quadratic expressions. Area models help us think about multiplying, factoring, and completing the square to find equivalent expressions. We modeled $(x+2)(x+5) = x^2 + 7x + 10$ as the area of a rectangle with sides of length $x + 2$ and $x + 5$. The various parts of the rectangle are shown in the diagram below:



Some people like to shortcut the area model a little bit to just have sections of area that correspond to the lengths of the sides. In this case, they might draw the following.

	x	$+5$
x	x^2	$5x$
$+2$	$2x$	10

$$= x^2 + 7x + 10$$

1. What is the property that all of these models are based upon?

2. Now that you've been reminded of the happy past, you are ready to use the strategy of your choice to find equivalent expressions for each of the following:

a) $(x + 3)(x + 4)$ b) $(x + 7)(x - 2)$

Maybe now you remember some of the different forms for quadratic expressions—factored form and standard form. These forms exist for all polynomials, although as the powers get higher, the algebra may get a little trickier. In standard form polynomials are written so that the terms are in order with the highest-powered term first, and then the lower-powered terms. Some examples:

Quadratic: $x^2 - 3x + 8$ or $x^2 - 9$

Cubic: $2x^3 + x^2 - 7x - 10$ or $x^3 - 2x^2 + 15$

Quartic: $x^4 + x^3 + 3x^2 - 5x + 4$

Hopefully, you also remember that you need to be sure that each term in the first factor is multiplied by each term in the second factor and the like terms are combined to get to standard form. You can use area models, boxes, or mnemonics like FOIL (first, outer, inner, last) to help you organize, or you can just check every time to be sure that you've got all the combinations. It can get more challenging with higher-powered polynomials, but the principal is the same because it is based upon the mighty Distributive Property.

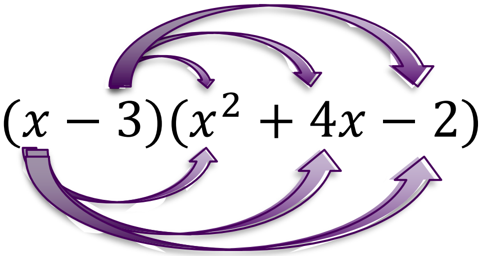
3. Tia's favorite strategy for multiplying polynomials is to make a box that fits the two factors. She sets it up like this: $(x+2)(x^2-3x+5)$

	x^2	$-3x$	$+5$
x			
$+2$			

Try using Tia's box method to multiply these two factors together and then combining like terms to get a polynomial in standard form.

4. Try checking your answer by graphing the original factored polynomial, $(x+2)(x^2-3x+5)$ and then graphing the polynomial that is your answer. If the graphs are the same, you are right because the two expressions are equivalent! If they are not the same, go back and check your work to make the corrections.

5. Tehani's favorite strategy is to connect the terms he needs to multiply in order like this:



Try multiplying using Tehani's strategy and then check your work by graphing. Make any corrections you need and figure out why they are needed so that you won't make the same mistake twice!

6. Use the strategy of your choice to multiply each of the following expressions. Check your work by graphing and make any needed corrections.

a) $(x+5)(x^2-x-3)$

b) $(x-2)(2x^2+6x+1)$

c) $(x+2)(x-2)(x+3)$

When graphing, it is often useful to have a perfect square quadratic or a perfect cube. Sometimes it is also useful to have these functions written in standard form. Let's try re-writing some related expressions to see if we can see some useful patterns.

7. Multiply and simplify both of the following expressions using the strategy of your choice:

a) $f(x) = (x + 1)^2$

b) $f\ (x) = (x + 1)^3$

Check your work by graphing and make any corrections needed.

8. Some enterprising young mathematician noticed a connection between the coefficients of the terms in the polynomial and the number pattern known as Pascal's Triangle. Put your answers from problem 5 into the table. Compare your answers to the numbers in Pascal's Triangle below and describe the relationship you see.

$(x + 1)^0$	1	1
$(x + 1)^1$	$x + 1$	1 1
$(x + 1)^2$		1 2 1
$(x + 1)^3$		1 3 3 1
$(x + 1)^4$		

9. It could save some time on multiplying the higher power polynomials if we could use Pascal's Triangle to get the coefficients. First, we would need to be able to construct our own Pascal's Triangle and add rows when we need to. Look at Pascal's Triangle and see if you can figure out how to get the next row using the terms from the previous row. Use your method to find the terms in the next row of the table above.

10. Now you can check your Pascal's Triangle by multiplying out $(x + 1)^4$ and comparing the coefficients. Hint: You might want to make your job easier by using your answers from #7 in some way. Put your answer in the table above.

11. Make sure that the answer you get from multiplying $(x+1)^4$ and the numbers in Pascal's Triangle match, so that you're sure you've got both answers right. Then describe how to get the next row in Pascal's Triangle using the terms in the previous row.

12. Complete the next row of Pascal's Triangle and use it to find the standard form of $(x+1)^5$. Write your answers in the table on #6.

13. Pascal's Triangle wouldn't be very handy if it only worked to expand powers of $x+1$. There must be a way to use it for other expressions. The table below shows Pascal's Triangle and the expansion of $x+a$.

$(x+a)^0$	1	1
$(x+a)^1$	$x+a$	1 1
$(x+a)^2$	$x^2+2ax+a^2$	1 2 1
$(x+a)^3$	$x^3+\ 3ax^2+3a^2x+a^3$	1 3 3 1
$(x+a)^4$	$x^4+4ax^3+\ 6a^2x^2+3a^3x+a^4$	1 4 6 4 1

What do you notice about what happens to the a in each of the terms in a row?

14. Use the Pascal's Triangle method to find standard form for $(x+2)^3$. Check your answer by multiplying.

15. Use any method to write each of the following in standard form:

a) $(x+3)^3$

b) $(x-2)^3$

c) $(x+5)^4$

READY, SET, GO! Name Period Date

READY

Topic: Recalling the meaning of division

1. Given: $f(x) = (x+7)(2x-3) \text{ and } g(x) = (x+7)$. Find $g(x)\overline{)f(x)}$.

2. Given: $f(x) = (5x+7)(-3x+11) \text{ and } g(x) = (-3x+11)$. Find $g(x)\overline{)f(x)}$

3. Given: $f(x) = (x+2)(x^2+3x+2) \text{ and } g(x) = (x+2)$ Find $g(x)\overline{)f(x)}$

4. Given: $f(x) = (5x-3)(x^2-11x-9) \text{ and } g(x) = (5x-3) \text{ and } h(x) = (x^2-11x-9)$.

   a.) Find $g(x)\overline{)f(x)}$ b.) Find $h(x)\overline{)f(x)}$

5. Given: $f(x) = (5x-6)(2x^2-5x+3) \text{ and } g(x) = (x-1) \text{ and } h(x) = (2x-3)$.

   a.) Find $g(x)\overline{)f(x)}$ b.) Find $h(x)\overline{)f(x)}$

SET

Topic: Multiplying polynomials

Multiply. Write your answers in standard form.

6. $(a+b)(a+b)$

7. $(x-3)(x^2+3x+9)$

8. $(x-5)(x^2+5x+25)$

9. $(x+1)(x^2-x+1)$

10. $(x+7)(x^2-7x+49)$

11. $(a-b)(a^2+ab+b^2)$

$(x+a)^0$	1	1
$(x+a)^1$	$x+a$	1 1
$(x+a)^2$	$x^2+2ax+a^2$	1 2 1
$(x+a)^3$	$x^3+3ax^2+3a^2x+a^3$	1 3 3 1
$(x+a)^4$	$x^4+4ax^3+6a^2x^2+3a^3x+a^4$	1 4 6 4 1

Use the table above to write each of the following in standard form.

12. $(x+1)^5$

13. $(x-5)^3$

14. $(x-1)^4$

15, $(x+4)^3$

16. $(x+2)^4$

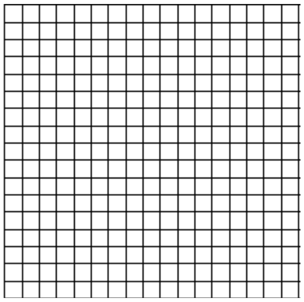
17. $(3x+1)^3$

GO

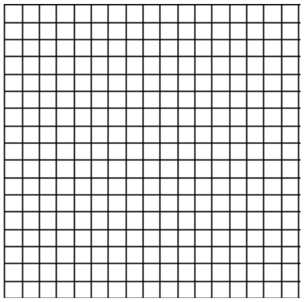
Topic: Examining transformations on different types of functions

Graph the following functions.

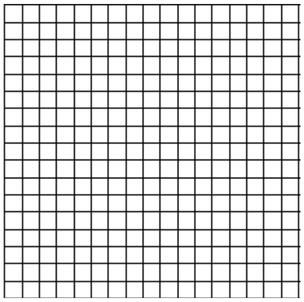
18. $g(x) = x + 2$



19. $h(x) = x^2 + 2$



20. $f(x) = 2^x + 2$



21. $g(x) = 3(x-2)$

22. $h(x) = 3(x-2)^2$

23. $f(x) = 3\sqrt{x-2}$

24. $g(x) = \frac{1}{2}(x-1) - 2$

25. $h(x) = \frac{1}{2}(x-1)^2 - 2$

26. $f(x) = |x-1| - 2$

3.5 Divide And Conquer

A Solidify Understanding Task

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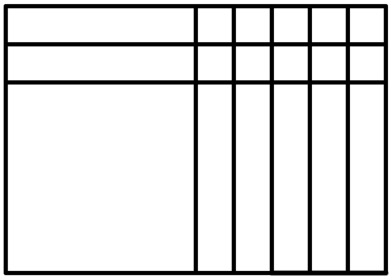
We've seen how numbers and polynomials relate in addition, subtraction, and multiplication. Now we're ready to consider division.

Division, you say? Like, long division? Yup, that's what we're talking about. Hold the judgment! It's actually pretty cool.

As usual, let's start by looking at how the operation works with numbers. Since division is the inverse operation of multiplication, the same models should be useful. The area model that we used with multiplication is also used with division. When we were using area models to factor a quadratic expression, we were actually dividing.

Let's brush up on that a bit.

1. The area model for $x^2 + 7x + 10$ is shown below:



Use the area model to write $x^2 + 7x + 10$ in factored form.

2. We also used number patterns to factor without drawing the area model. Use any strategy to factor the following quadratic polynomials:

a) $x^2 + 7x + 12$	b) $x^2 + 2x - 15$

c) $x^2 - 11x + 24$	d) $x^2 - 5x - 36$

Factoring works great for quadratics and a few special cases of other polynomials. Let's look at a more general version of division that is a lot like what we do with numbers. Let's say we want to divide 1452 by 12. If we write the analogous polynomial division problem, it would be: $(x^3 + 4x^2 + 5x + 2) \div (x + 2)$.

Let's use the division process for numbers to create a division process for polynomials. (Don't panic—in many ways it's easier with polynomials than numbers!)

Step 1: Start with writing the problem as long division. The polynomial needs to have the terms written in descending order. If there are any missing powers, it's easier if you leave a little space for them.

$$12\overline{)1452} \qquad\qquad x+2\overline{)x^3+4x^2+5x+2}$$

Step 2: Determine what you could multiply the divisor by to get the first term of the dividend.

$$\begin{array}{r} 1 \\ 12\overline{)1452} \end{array} \qquad\qquad \begin{array}{r} x^2 \\ x+2\overline{)x^3+4x^2+5x+2} \end{array}$$

Step 3: Multiply and put the result below the dividend.

$$\begin{array}{r} 1 \\ 12\overline{)1452} \\ \underline{-1200} \end{array} \qquad\qquad \begin{array}{r} x^2 \\ x+2\overline{)x^3+4x^2+5x+2} \\ -(\underline{x^3+2x^2}) \end{array}$$

Step 4: Subtract. (It helps to keep the signs straight if you change the sign on each term and add on the polynomial.)

$$\begin{array}{r} 1 \\ 12\overline{)1452} \\ \underline{-1200} \\ 252 \end{array}$$

$$\begin{array}{r} x^2 \\ x+2\overline{)x^3+4x^2+5x+2} \\ +(\underline{-x^3-2x^2\qquad\qquad}) \\ 2x^2+5x+2 \end{array}$$

Step 5: Repeat the process with the number or expression that remains in the dividend.

$$\begin{array}{r} 12 \\ 12\overline{)1452} \\ \underline{-1200} \\ 252 \\ -\underline{240} \\ 12 \end{array}$$

$$\begin{array}{r} x^2+2x \\ x+2\overline{)x^3+4x^2+5x+2} \\ +(\underline{-x^3-2x^2\qquad\qquad}) \\ 2x^2+5x+2 \\ -\underline{(2x^2+4x\qquad)} \\ x+2 \end{array}$$

Step 6: Keep going until the number or expression that remains is smaller than the divisor.

$$\begin{array}{r} 121 \\ 12\overline{)1452} \\ \underline{-1200} \\ 252 \\ -\underline{240} \\ 12 \\ -\underline{12} \\ 0 \end{array}$$

$$\begin{array}{r} x^2+2x+1 \\ x+2\overline{)x^3+4x^2+5x+2} \\ +(\underline{-x^3-2x^2\qquad\qquad}) \\ 2x^2+5x+2 \\ -\underline{(2x^2+4x\qquad)} \\ x+2 \\ -\underline{(x+2)} \\ 0 \end{array}$$

In this case, 121 divided by 12 leaves no remainder, so we would say that 12 is a factor of 121. Similarly, since (x^3+4x^2+5x+2) divided by $(x+2)$ leaves no remainder, we would say that $(x+2)$ is a factor of (x^3+4x^2+5x+2).

Polynomial division doesn't always match up perfectly to an analogous whole number problem, but the process is always the same. Let's try it.

3. Use long division to determine if $(x - 1)$ a factor of $(x^3 - 3x^2 - 13x + 15)$. Don't worry: the steps for the division process are below:

a) Write the problem as long division.
b) What do you have to multiply x by to get x^3? Write your answer above the bar.
c) Multiply your answer from step b by $(x - 1)$ and write your answer below the dividend.
d) Subtract. Be careful to subtract each term. (You might want to change the signs and add.)
e) Repeat steps a-d until the expression that remains is less than $(x - 1)$.

We hope you survived the division process. Is $(x - 1)$ a factor of $(x^3 - 3x^2 - 13x + 15)$? ________

4. Try it again. Use long division to determine if $(2x + 3)$ is a factor of $2x^3 + 7x^2 + 2x + 9$. No hints this time. You can do it!

When dividing numbers, there are several ways to deal with the remainder. Sometimes, we just write it as the remainder, like this:

$$\begin{array}{r} 8r.1 \\ 3\overline{)\ \ 25} \end{array} \quad \text{because } 3(8) + 1 = 25$$

You may remember also writing the remainder as a fraction like this:

$\begin{array}{r} 8\ {}^{1}\!/_{3} \\ 3\overline{)\ \ 25} \end{array}$ because $3\left(8\frac{1}{3}\right) = 25$

We do the same things with polynomials.

Maybe you found that $(2x^3 + 7x^2 + 2x + 9) \div (2x + 3) = \ (x^2 + 2x - 2)\ r. 15$. (We sure hope so.) You can use it to write two multiplication statements:

$$(2x + 3)(x^2 + 2x - 2) + 15 = (2x^3 + 7x^2 + 2x + 9)$$

and

$$(2x + 3)(x^2 + 2x - 2 + \frac{15}{2x + 3}) = (2x^3 + 7x^2 + 2x + 9)$$

5. Divide each of the following polynomials. Write the two multiplication statements that go with your answers if there is a remainder. Write only one multiplication statement if the divisor is a factor. Use graphing technology to check your work and make the necessary corrections.

	a) $(x^3 + 6x^2 + 13x + 12) \div (x + 3)$	b) $(x^3 - 4x^2 + 2x + 5) \div (x - 2)$
Multiplication statements:		

	c) $(6x^3 - 11x^2 - 4x + 5) \div (2x - 1)$	d) $(x^4 - 23x^3 + 49x + 4) \div (x^2 + x + 2)$
Multiplication statements:		

READY, SET, GO! Name Period Date

READY

Topic: Solving linear equations

Solve for x.

1. $5x + 13 = 48$
2. $\frac{1}{3}x - 8 = 0$
3. $-4 - 9x = 0$
4. $x^2 - 16 = 0$
5. $x^2 + 4x + 3 = 0$
6. $x^2 - 5x + 6 = 0$
7. $(x + 8)(x + 11) = 0$
8. $(x - 5)(x - 7) = 0$
9. $(3x - 18)(5x - 10) = 0$

SET

Topic: Dividing polynomials

Divide each of the following polynomials. Write only one multiplication statement if the divisor is a factor. Write the two multiplication statements that go with your answers if there is a remainder.

10. $(x+1)\overline{\big)x^3-3x^2+6x+11}$

11. $(x-5)\overline{\big)x^3-9x^2+23x-15}$

Multiplication statement(s)

Multiplication statement(s)

12. $(2x-1)\overline{)2x^3+15x^2-34x+13}$

13. $(x+4)\overline{)x^3+13x^2+26x-25}$

Multiplication statement(s)

Multiplication statement(s)

14. $(x+7)\overline{)x^3-8x^2-111x+10}$

15. $(3x-4)\overline{)3x^3+23x^2+6x-28}$

Multiplication statement(s)

Multiplication statement(s)

GO

Topic: Describing the features of a variety of functions

Graph the following functions. Then identify the key features of the functions. Include domain, range, intervals where the function is increasing/decreasing, intercepts, maximum/minimum, and end behavior.

16. $f(x) = x^2 - 9$

domain: range:

increasing: decreasing:

y-intercept: x-intercept(s):

17. $f(n-1) = f(n) + 3;\ f(1) = 4$

domain: range:

increasing: decreasing:

y-intercept: x-intercept(s):

18. $f(x) = \sqrt{x-3} + 1$

domain: range:

increasing: decreasing:

y-intercept: x-intercept(s):

19. $f(x) = log_2 x - 1$

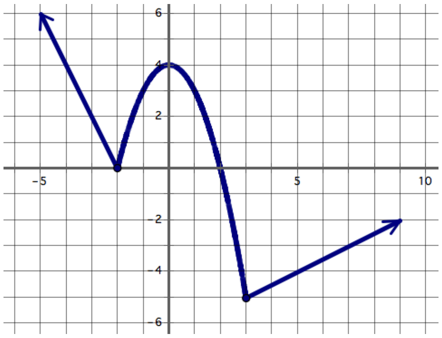
domain: range:

increasing: decreasing:

y-intercept: x-intercept(s):

Identify the key features of the graphed functions.

20.

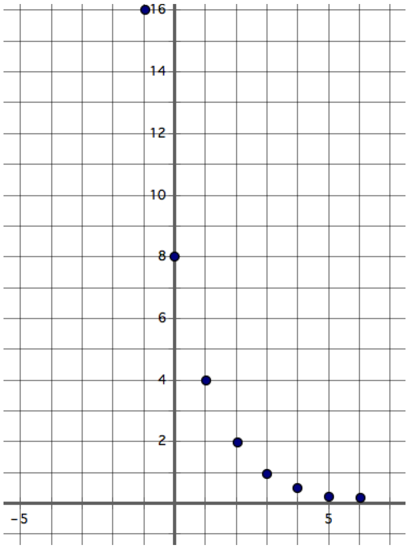


domain: range:

increasing: decreasing:

y-intercept: x-intercept(s):

21.



domain: range:

increasing: decreasing:

y-intercept: x-intercept(s):

3.6 Sorry, We're Closed

A Practice Understanding Task

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Now that we have compared operations on polynomials with operations on whole numbers it's time to generalize about the results. Before we go too far, we need a technical definition of a polynomial function. Here it is:

A polynomial function has the form:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

where $a_n, a_{n-1}, \ldots a_1, a_0$ are real numbers and n is a nonnegative integer. In other words, a polynomial is the sum of one or more monomials with real coefficients and nonnegative integer exponents. The degree of the polynomial function is the highest value for n where a_n is not equal to 0.

1. The following examples and non-examples will help you to see the important implications of the definition of a polynomial function. For each pair, determine what is different between the example of a polynomial and the non-example that is not a polynomial.

These **are** polynomials:	These **are not** polynomials:
a) $f(x) = x^3$	b) $g(x) = 3^x$
How are a and b different?	
c) $f(x) = 2x^2 + 5x - 12$	d) $g(x) = \frac{2x^2}{x^2-3x+2}$
How are c and d different?	
e) $f(x) = -x^3 + 3x^2 - 2x - 7$	f) $g(x) = x^3 + 3x^2 - 2x + 10x^{-1} - 7$
How are e and f different?	
h) $f(x) = \frac{1}{2}x$	i) $g(x) = \frac{1}{2x}$
How are h and i different?	
j) $f(x) = x^2$	k) $g(x) = x^{\frac{1}{2}}$
How are j and k different?	

2. Based on the definition and the examples above, how can you tell if a function is a polynomial function?

Maybe you have noticed in the past that when you add two even numbers, the answer you get is always an even number. Mathematically, we say that the set of even numbers is **closed** under addition. Mathematicians are interested in results like this because it helps us to understand how numbers or functions of a particular type behave with the various operations.

3. You can try it yourself: Is the set of odd numbers closed under multiplication? In other words, if you multiply two odd numbers together will you get an odd number? Explain.

If you find any two odd numbers that have an even product, then you would say that odd numbers are not closed under multiplication. Even if you have a number of examples that support the claim, if you can find one **counterexample** that contradicts the claim, then the claim is false.

Consider the following claims and determine whether they are true or false. If a claim is true, give a reason with at least two examples that illustrate the claim. Your examples can include any representation you choose. If the claim is false, give a reason with one counterexample that proves the claim to be false.

4. The set of whole numbers is closed under addition.

5. The sum of a quadratic function and a linear function is a cubic function.

6. The sum of a linear function and an exponential function is a polynomial.

7. The set of polynomials is closed under addition.

8. The set of whole numbers is closed under subtraction.

9. The set of integers is closed under subtraction.

10. A quadratic function subtracted from a cubic function is a cubic function.

11. A linear function subtracted from a linear function is a polynomial function.

12. A cubic function subtracted from a cubic function is a cubic function.

13. The set of polynomial functions is closed under subtraction.

14. The product of two linear functions is a quadratic function.

15. The set of integers is closed under multiplication.

16. The set of polynomials is closed under multiplication.

17. The set of integers is closed under division.

18. A cubic function divided by a linear function is a quadratic function.

19. The set of polynomial functions is closed under division.

20. Write two claims of your own about polynomials and use examples to demonstrate that they are true.
Claim #1:

Claim #2:

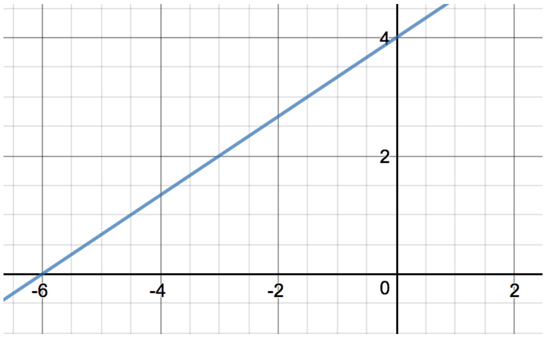
READY, SET, GO! Name Period Date

READY

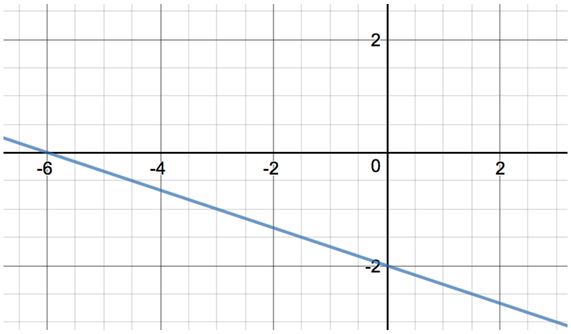
Topic: Connecting the zeros of a function to the solution of the equation

When we solve equations, we often set the equation equal to zero and then find the value of x. Another way to say this is "find when $f(x) = 0$." That's why we call solutions to equations the zeros of an equation. Find the zeros for the given equations. Then mark the solution(s) as a point on the graph of the equation.

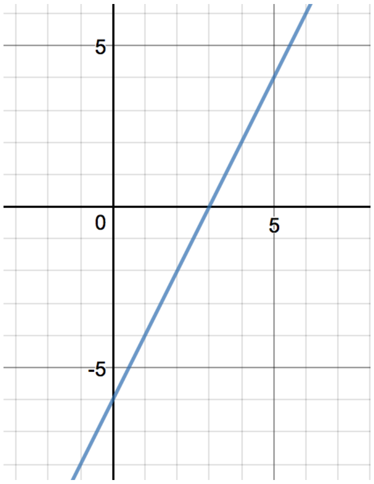
1. $f(x) = \frac{2}{3}x + 4$



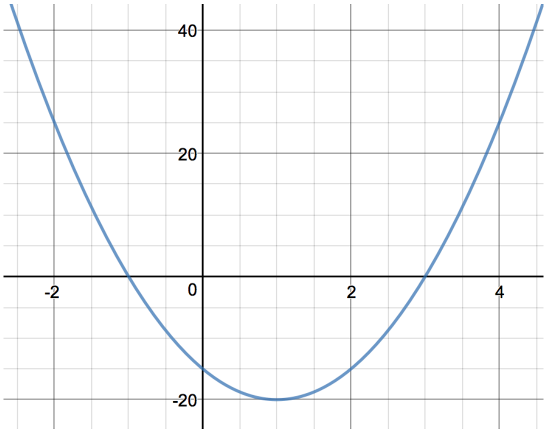
2. $g(x) = -\frac{1}{3}x - 2$



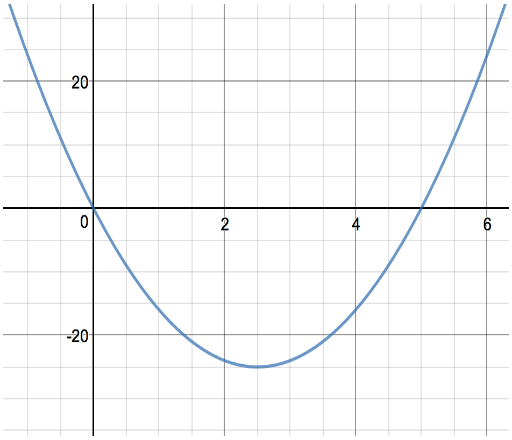
3. $h(x) = 2x - 6$



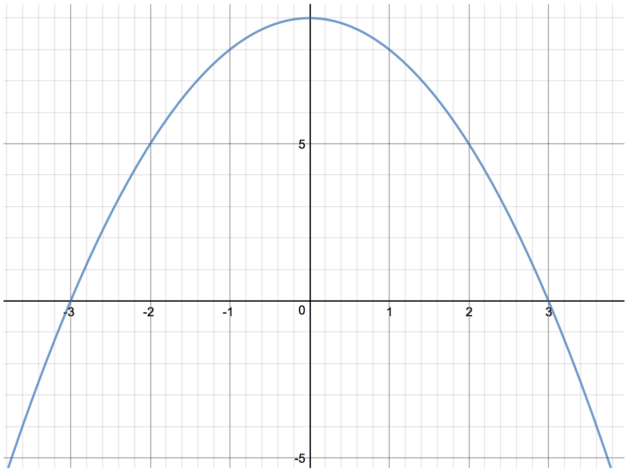
4. $p(x) = 5x^2 - 10x - 15$



5. $q(x) = 4x^2 - 20x$



6. $d(x) = -x^2 + 9$



SET

Topic: Exploring closed mathematical number sets

Identify the following statements as *sometimes* true, *always* true, or *never* true. If your answer is *sometimes* true, give an example of when it's true and an example of when it's not true. If it's *never* true, give a counter-example.

7. The product of a whole number and a whole number is an integer.

8. The quotient of a whole number divided by a whole number is a whole number.

9. The set of integers is **closed** under division.

10. The difference of a linear function and a linear function is an integer.

11. The difference of a linear function and a quadratic function is a linear function.

12. The product of a linear function and a linear function is a quadratic function.

13. The sum of a quadratic function and a quadratic function is a polynomial function.

14. The product of a linear function and a quadratic function is a cubic function.

15. The product of three linear functions is a cubic function.

16. The set of polynomial functions is **closed** under addition.

GO

Topic: Identifying conjugate pairs

A **conjugate pair** is simply a pair of binomials that have the same numbers but differ by having opposite signs between them. For example $(a + b)$ and $(a - b)$ are conjugate pairs. You've probably noticed them when you've factored a quadratic expression that is the difference of two squares. **Example:** $x^2 - 25 = (x+5)(x-5)$. The two factors $(x+5)(x-5)$ are conjugate pairs.

The quadratic formula $\boldsymbol{x} = \frac{-\boldsymbol{b} \pm \sqrt{\boldsymbol{b}^2 - 4\boldsymbol{ac}}}{2\boldsymbol{a}}$ can generate both solutions to a quadratic equation because of the $\pm$ located in the numerator of the formula. When the $\sqrt{b^2 - 4ac}$ part of the formula generates an irrational number (e.g. $\sqrt{2}$) or an imaginary number (e.g. $2i$), the formula produces a pair of numbers that are conjugates. This is important because <u>this type of solution to a quadratic always comes in pairs.</u> **Example:** The conjugate of $(3 + \sqrt{2})$ *is* $(3 - \sqrt{2})$. The conjugate of $(-2i)$ is $(+2i)$. Think of it as $(0 - 2i)$ *and* $(0 + 2i)$. <u>Change only the sign between the two numbers.</u>

Write the conjugate of the given value.

17. $(8 + \sqrt{5})$

18. $(11 + 4i)$

19. $9i$

20. $-5\sqrt{7}$

21. $(2 - 13i)$

22. $(-1 - 2i)$

23. $(-3 + 5\sqrt{2})$

24. $-4i$

3.7 Building Strong Roots

A Solidify Understanding Task

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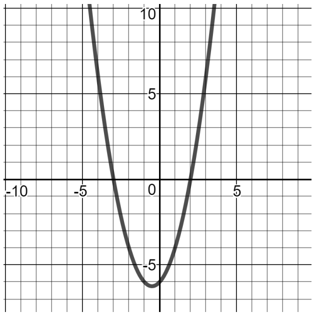
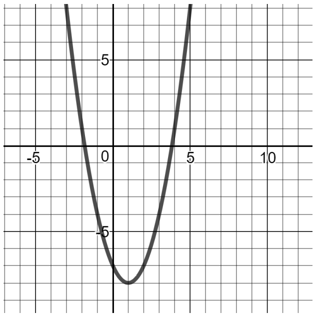
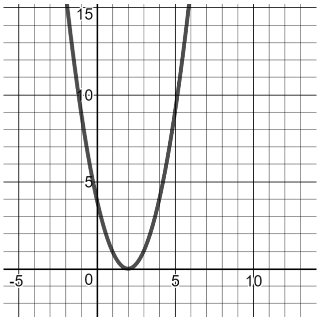
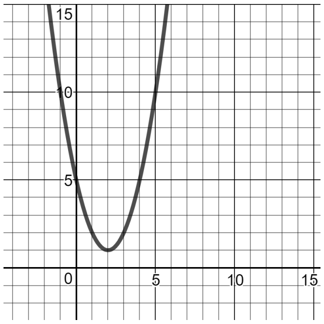
When working with quadratic functions, we learned the Fundamental Theorem of Algebra:
An n^{th} degree polynomial function has n roots.
In this task, we will be exploring this idea further with other polynomial functions.

First, let's brush up on what we learned about quadratics. The equations and graphs of four different quadratic equations are given below. Find the roots for each and identify whether the roots are real or imaginary.

1.

a) $f(x) = x^2 + x - 6$  	b) $g(x) = x^2 - 2x - 7$ 
Roots:	Roots:
Type of roots:	Type of roots:
c) $h(x) = x^2 - 4x + 4$  	d) $k(x) = x^2 - 4x + 5$ 
Roots:	Roots:
Type of roots:	Type of roots:

2. Did all of the quadratic functions have 2 roots, as predicted by the Fundamental Theorem of Algebra? Explain.

3. It's always important to keep what you've previously learned in your mathematical bag of tricks so that you can pull it out when you need it. What strategies did you use to find the roots of the quadratic equations?

4. Using your work from problem 1, write each of the quadratic equations in factored form. When you finish, check your answers by graphing, when possible, and make any corrections necessary.

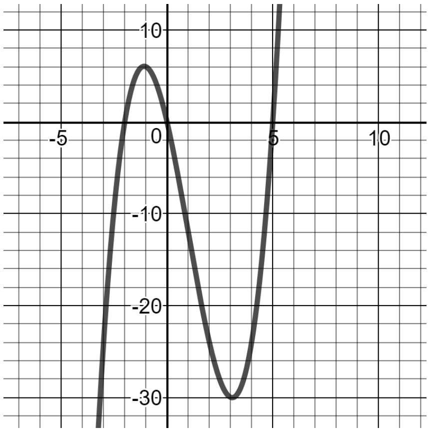
a) $f(x) = x^2 + x - 6$	b) $g(x) = x^2 - 2x - 7$
Factored form:	Factored form:
c) $h(x) = x^2 - 4x + 4$	d) $k(x) = x^2 - 4x + 5$
Factored form:	Factored form:

5. Based on your work in problem 1, would you say that roots are the same as x-intercepts? Explain.

6. Based on your work in problem 4, what is the relationship between roots and factors?

Now let's take a closer look at cubic functions. We've worked with transformations of $f(x) = x^3$, but what we've seen so far is just the tip of the iceberg. For instance, consider:

$g(x) = x^3 - 3x^2 - 10x$



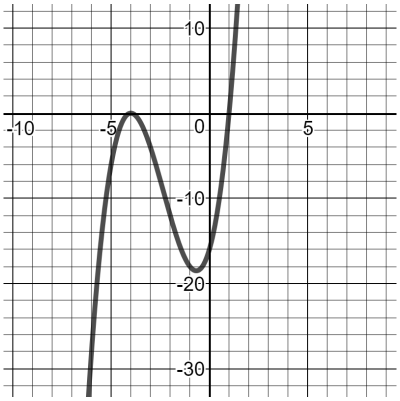
7. Use the graph to find the roots of the cubic function. Use the equation to verify that you are correct. Show how you have verified each root.

8. Write $g(x)$ in factored form. Verify that the factored form is equivalent to the standard form.

9. Are the results you found in #7 consistent with the Fundamental Theorem of Algebra? Explain.

Here's another example of a cubic function.

$f(x) = x^3 + 7x^2 + 8x - 16$

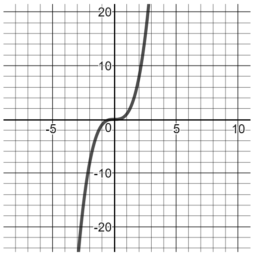


10. Use the graph to find the roots of the cubic function.

11. Write $f(x)$ in factored form. Verify that the factored form is equivalent to the standard form. Make any corrections needed.

12. Are the results you found in #10 consistent with the Fundamental Theorem of Algebra? Explain.

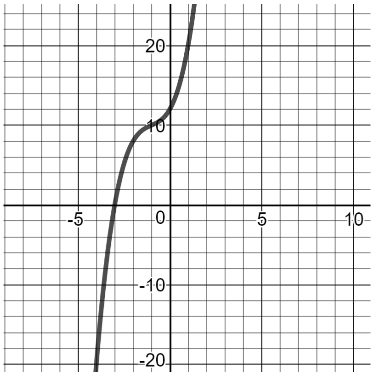
13. We've seen the most basic cubic polynomial function, $h(x) = x^3$ and we know its graph looks like this:



Explain how $h(x) = x^3$ is consistent with the Fundamental Theorem of Algebra.

14. Here is one more cubic polynomial function for your consideration. You will notice that it is given to you in factored form. Use the equation and the graph to find the roots of $p(x)$.

$$p(x) = (x + 3)(x^2 + 4)$$



15. Use the equation to verify each root. Show your work below.

16. Are the results you found in #14 consistent with the Fundamental Theorem of Algebra? Explain.

17. Explain how to find the factored form of a polynomial, given the roots.

18. Explain how to find the roots of a polynomial, given the factored form.

READY, SET, GO! Name Period Date

READY

Topic: Practicing long division on polynomials

Divide using long division. (These problems have no remainders. If you get one, try again.)

1. $(x+3)\overline{)5x^3+2x^2-45x-18}$

2. $(x-6)\overline{)x^3-x^2-44x+84}$

3. $(x-5)\overline{)3x^3-15x^2+12x-60}$

4. $(x+2)\overline{)x^4+6x^3+7x^2-6x-8}$

SET

Topic: Applying the Fundamental Theorem of Algebra

Predict the number of roots for each of the given polynomial equations. (Remember that the Fundamental Theorem of Algebra states: An n^{th} degree polynomial function has n roots.)

5. $a(x) = x^2 + 3x - 10$

6. $b(x) = x^3 + x^2 - 9x - 9$

7. $c(x) = -2x - 4$

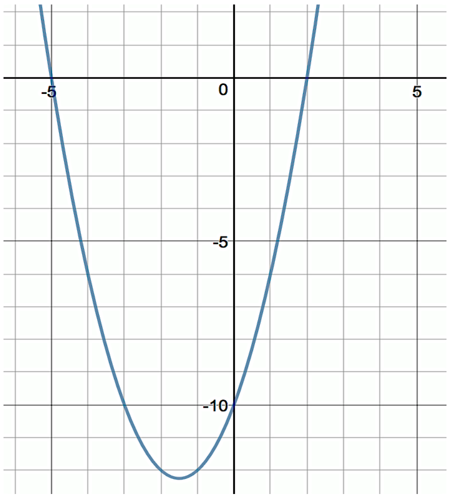
8. $d(x) = x^4 - x^3 - 4x^2 + 4x$

9. $f(x) = -x^2 + 6x - 9$

10. $g(x) = x^6 - 5x^4 + 4x^2$

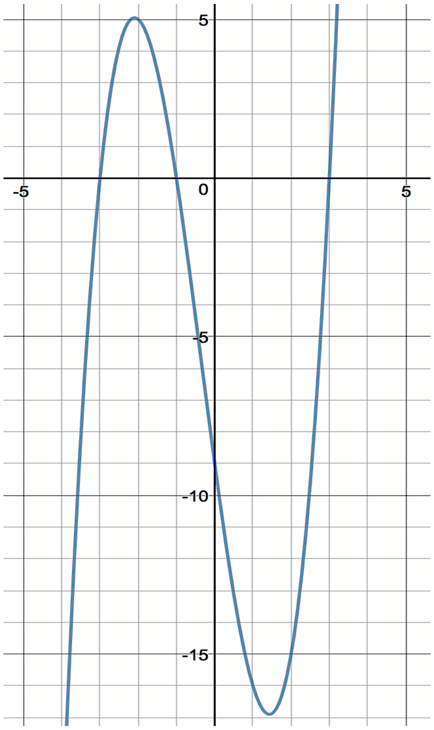
Below are the graphs of the polynomials from the previous page. Check your predictions. Then use the graph to help you write the polynomial in factored form.

11. $a(x) = x^2 + 3x - 10$



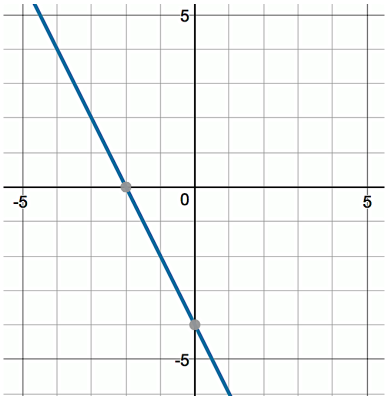
Factored form:

12. $b(x) = x^3 + x^2 - 9x - 9$



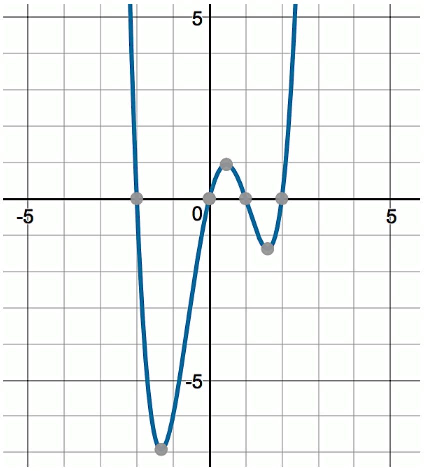
Factored form:

13. $c(x) = -2x - 4$



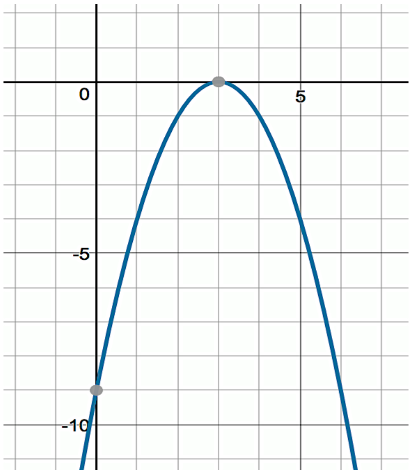
Factored form:

14. $d(x) = x^4 - x^3 - 4x^2 + 4x$



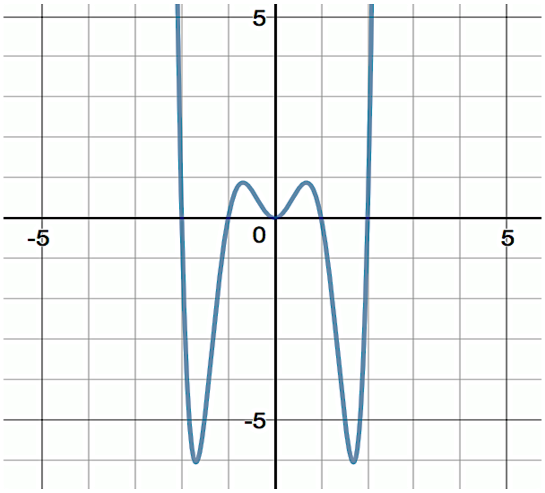
Factored form:

15. $f(x) = -x^2 + 6x - 9$



Factored form:

16. $g(x) = x^6 - 5x^4 + 4x^2$



Factored form:

17. The graphs of #15 and #16 don't seem to follow the Fundamental Theorem of Algebra, but there is something similar about each of the graphs. Explain what is happening at the point (3, 0) in #15 and at the point (0,0) in #16.

GO

Topic: Solving quadratic equations

Find the zeros for each equation using the quadratic formula.

18. $f(x) = x^2 + 20x + 51$

19. $f(x) = x^2 + 10x + 25$

20. $f(x) = 3x^2 + 12x$

21. $f(x) = x^2 - 11$

22. $f(x) = x^2 + x - 1$

23. $f(x) = x^2 + 2x + 3$

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3.8 Getting to the Root of the Problem

A Solidify Understanding Task

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In *3.7 Building Strong Roots,* we learned to predict the number of roots of a polynomial using the Fundamental Theorem of Algebra and the relationship between roots and factors. In this task, we will be working on how to find all the roots of a polynomial given in standard form.

Let's start by thinking again about numbers and factors.

1. If you know that 7 is a factor of 147, what would you do to find the prime factorization of 147? Explain your answer and show your process here:

2. How is your answer like a polynomial written in the form: $P(x) = (x-7)^2(x-3)$?

The process for finding factors of polynomials is exactly like the process for finding factors of numbers. We start by dividing by a factor we know and keep dividing until we have all the factors. When we get the polynomial broken down to a quadratic, sometimes we can factor it by inspection, and sometimes we can use our other quadratic tools like the quadratic formula.

Let's try it! For each of the following functions, you have been given one factor. Use that factor to find the remaining factors, the roots of the function, and write the function in factored form.

3. Function: $f(x) = x^3 + 3x^2 - 4x - 12$ Factor: $(x+3)$ Roots of function:

Factored form:

4. Function: $f(x) = x^3 + 6x^2 + 11x + 6$ Factor: $(x + 1)$ Roots of function:

Factored form:

5. Function: $f(x) = x^3 - 5x^2 - 3x + 15$ Factor: $(x - 5)$ Roots of function:

Factored form:

6. Function: $f(x) = x^3 + 3x^2 - 12x - 18$ Factor: $(x - 3)$ Roots of function:

Factored form:

7. Function: $f(x) = x^4 - 16$ Factor: $(x - 2)$ Roots of function:

Factored form:

8. Function: $f(x) = x^3 - x^2 + 4x - 4$ Factor: $(x - 2i)$ Roots of function:

Factored form:

9. Is it possible for a polynomial with real coefficients to have only one imaginary root? Explain.

10. Based on the Fundamental Theorem of Algebra and the polynomials that you have seen, make a table that shows all the number of roots and the possible combinations of real and imaginary roots for linear, quadratic, cubic, and quartic polynomials.

READY, SET, GO! Name Period Date

READY

Topic: Ordering numbers from least to greatest

Order the numbers from least to greatest.

1. 100^3 $\sqrt{100}$ $log_2 100$ 100 2^{10}

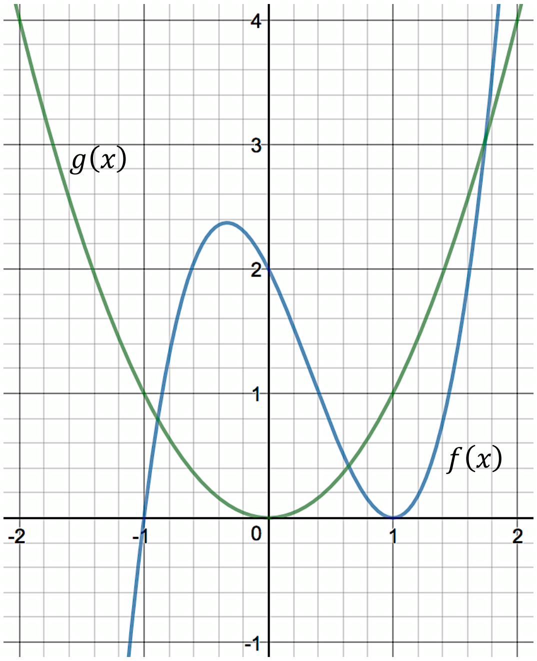
2. 2^{-1} $-\sqrt{100}$ $log_2\left(\frac{1}{8}\right)$ 0 $(-2)^1$

3. 2^0 $\sqrt{25}$ $log_2 8$ $2(x^0), x \neq 0$ $(2)^{-\frac{1}{2}}$

4. $log_3 3^3$ $log_5 5^{-2}$ $log_6 6^0$ $log_4 4^{-1}$ $log_2 2^3$

Refer to the given graph to answer the questions.

Insert $>, <, or =$ in each statement to make it true.



5. $f(0)$ ________ $g(0)$

6. $f(2)$ ________ $g(2)$

7. $f(-1)$ ________ $g(-1)$

8. $f(1)$ ________ $g(-1)$

9. $f(5)$ ________ $g(5)$

10. $f(-2)$ ________ $g(-2)$

SET

Topic: Finding the roots and factors of a polynomial

Use the given root to find the remaining roots. Then write the function in factored form.

Function	Roots	Factored form
11. $f(x) = x^3 - 13x^2 + 52x - 60$	$x = 5$	
12. $g(x) = x^3 + 6x^2 - 11x - 66$	$x = -6$	
13. $p(x) = x^3 + 17x^2 + 92x + 150$	$x = -3$	
14. $q(x) = x^4 - 6x^3 + 3x^2 + 12x - 10$	$x = \sqrt{2}$	

GO

Topic: Using the distributive property to multiply complex expressions

Multiply using the distributive property. Simplify. Write answers in standard form.

15. $\left(x-\sqrt{13}\right)\left(x+\sqrt{13}\right)$

16. $\left(x-3\sqrt{2}\right)\left(x+3\sqrt{2}\right)$

17. $(x-4+2i)(x-4-2i)$

18. $(x+5+3i)(x+5-3i)$

19. $(x-1+i)(x-1-i)$

20. $\left(x+10-\sqrt{2}\,i\right)\left(x+10+\sqrt{2}\,i\right)$

3.9 Is This The End?

A Solidify Understanding Task

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In previous mathematics courses, you have compared and analyzed growth rates of polynomial (mostly linear and quadratic) and exponential functions. In this task, we are going to analyze rates of change and end behavior by comparing various expressions.

Part I: Seeing patterns in **end behavior**

1. In as many ways as possible, compare and contrast linear, quadratic, cubic, and exponential functions.

2. Using the graph provided, write the following functions vertically, from **greatest to least for $\boldsymbol{x = 0}$**. Put the function with the greatest value on top and the function with the smallest value on the bottom. Put functions with the same values at the same level. An example, $l(x) = x^7$, has been placed on the graph to get you started.

$f(x) = 2^x$	$p(x) = x^3 + x^2 - 4$	$g(x) = x^2 - 20$
$h(x) = x^5 - 4x^2 + 1$	$k(x) = x + 30$	$m(x) = x^4 - 1$
$r(x) = x^5$	$n(x) = \left(\frac{1}{2}\right)^x$	$q(x) = x^6$

3. What determines the value of a polynomial function at $x = 0$? Is this true for other types of functions?

4. Write the same expressions on the graph in order from **greatest to least** when x represents a very large number (this number is so large, so we say that it is approaching positive infinity). If the value of the function is positive, put the function in quadrant 1. If the value of the function is negative, put the function in quadrant IV. An example has been placed for you.

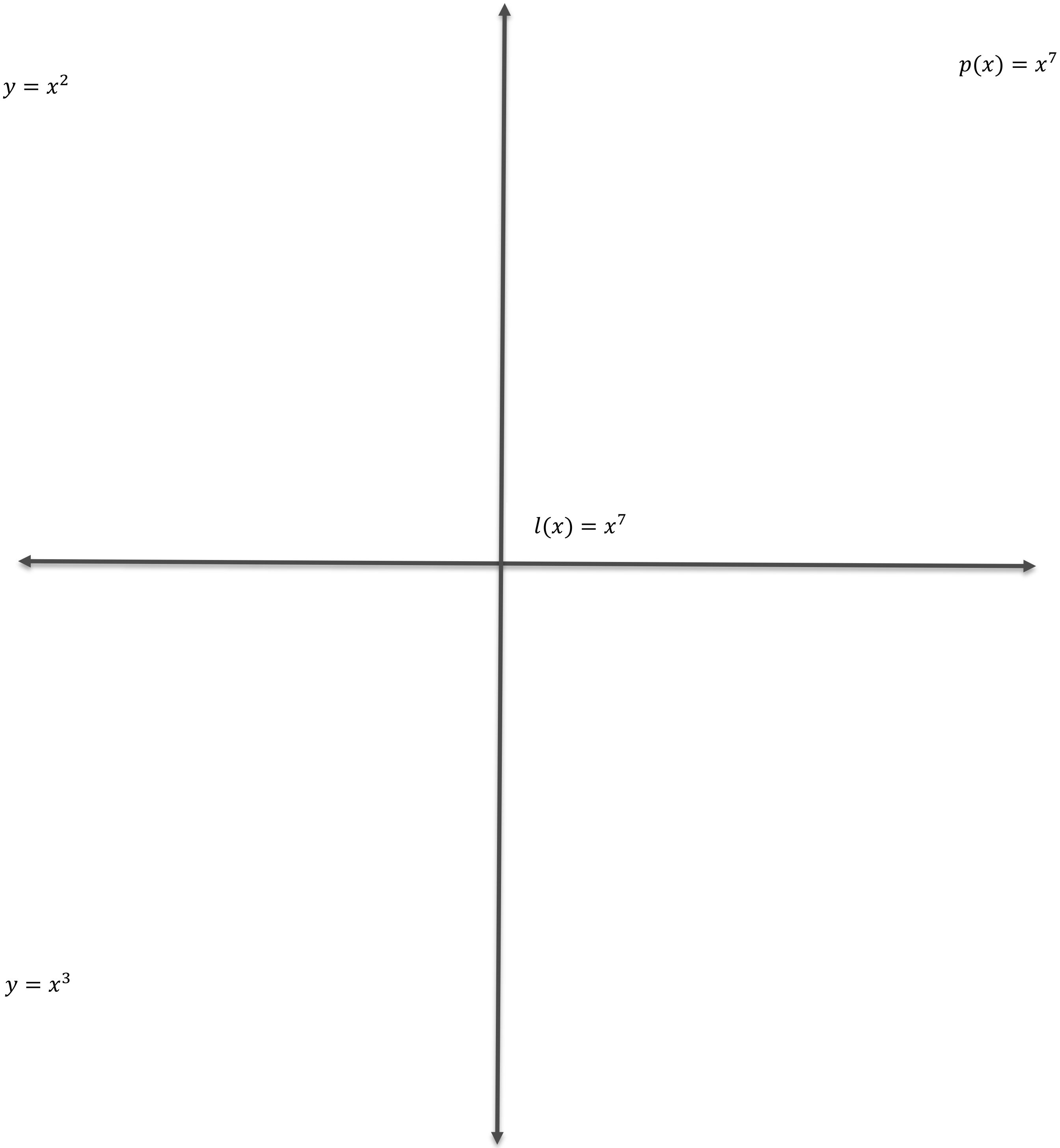
5. What determines the end behavior of a polynomial function for very large values of x?

6. Write the same functions in order from **greatest to least** when x represents a number that is approaching negative infinity. If the value of the function is positive, place it in Quadrant II, if the value of the function is negative, place it in Quadrant III. An example is shown on the graph.

7. What patterns do you see in the polynomial functions for x values approaching negative infinity? What patterns do you see for exponential functions? Use graphing technology to test these patterns with a few more examples of your choice.

8. How would the end behavior of the polynomial functions change if the lead terms were changed from positive to negative?

$\boldsymbol{x \to -\infty}$ $\boldsymbol{x = 0}$ $\boldsymbol{x \to \infty}$



Part II: Using end behavior patterns

For each situation:

- Determine the function type. If it is a polynomial, state the degree of the polynomial and whether it is an even degree polynomial or an odd degree polynomial.
- Describe the end behavior based on your knowledge of the function. Use the format: As $x \to -\infty, f(x) \to$ _____ *and* *as* $x \to \infty\ f(x) \to$ _____

1. $f(x) = 3 + 2x$

Function type:

End behavior: As $x \to -\infty, f(x) \to$ _____

End behavior: As $x \to \infty, f(x) \to$ _____

2. $f(x) = x^4 - 16$

Function type:

End behavior: As $x \to -\infty, f(x) \to$ _____

End behavior: As $x \to \infty, f(x) \to$ _____

3. $f(x) = 3^x$

Function type:

End behavior: As $x \to -\infty, f(x) \to$ _____

End behavior: As $x \to \infty, f(x) \to$ _____

4. $f(x) = x^3 + 2x^2 - x + 5$

Function type:

End behavior: As $x \to -\infty, f(x) \to$ _____

End behavior: As $x \to \infty, f(x) \to$ _____

5. $f(x) = -2x^3 + 2x^2 - x + 5$

Function type:

End behavior: As $x \to -\infty, f(x) \to$ _____

End behavior: As $x \to \infty, f(x) \to$ _____

6. $f(x) = log_2 x$

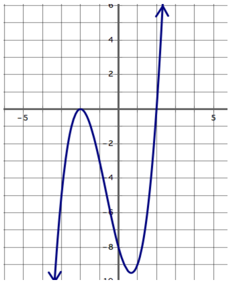
Function type:

End behavior: As $x \to -\infty, f(x) \to$ _____

End behavior: As $x \to \infty, f(x) \to$ _____

Use the graphs below to describe the end behavior of each function by completing the statements.

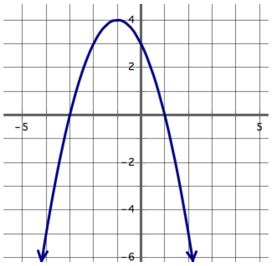
7.



End behavior: As $x \to -\infty, f(x) \to$ _____

End behavior: As $x \to \infty, f(x) \to$ _____

8.



End behavior: As $x \to -\infty, f(x) \to$ _____

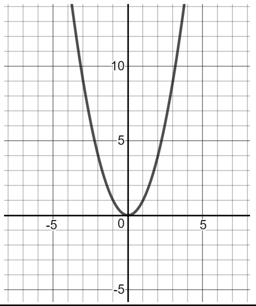
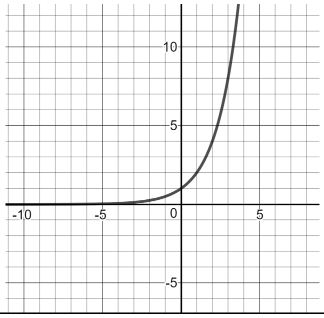
End behavior: As $x \to \infty, f(x) \to$ _____

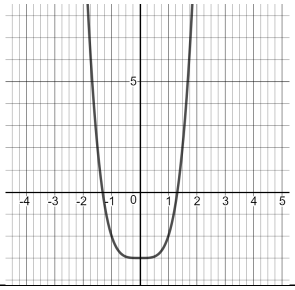
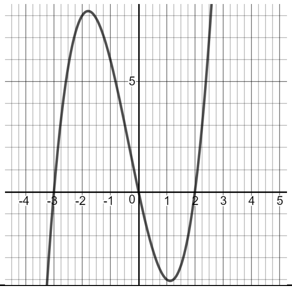
9. How does the end behavior for quadratic functions connect with the number and type of roots for these functions? How does the end behavior for cubic functions connect with the number and type of roots for cubic functions?

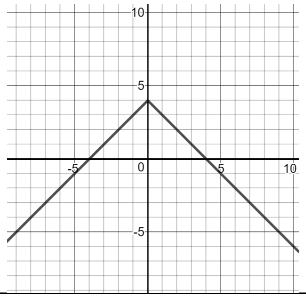
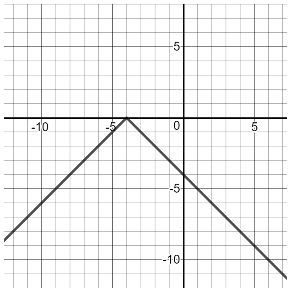
Part III: Even and Odd Functions

Some functions that are not polynomials may be categorized as even functions or odd functions. When mathematicians say that a function is an even function, they mean something very specific.

1. Let's see if you can figure out what the definition of an even function is with these examples:

Even function: $f(x) = x^2$ 	Not an even function: $g(x) = 2^x$ 
Differences:	

Even function: $f(x) = x^4 - 3$ 	Not an even function: $g(x) = x(x+3)(x-2)$ 
Differences:	

<table>
<tr><td>Even function:
$f(x) = -|x| + 4$
 </td><td>Not an even function:
$g(x) = -|x + 4|$
 </td></tr>
<tr><td colspan="2">Differences:</td></tr>
<tr><td>Even function:
$f(2) = 5 \; and \; f(-2) = 5$</td><td>Not an even function:
$g(2) = 3 \; and \; g(-2) = 5$</td></tr>
<tr><td colspan="2">Differences:</td></tr>
</table>

2. What do you observe about the characteristics of an even function?

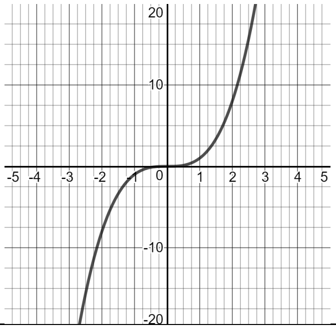
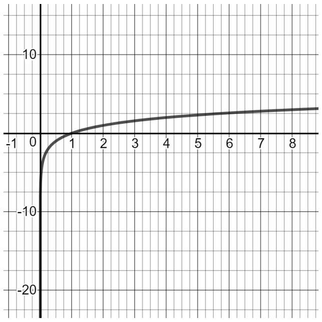
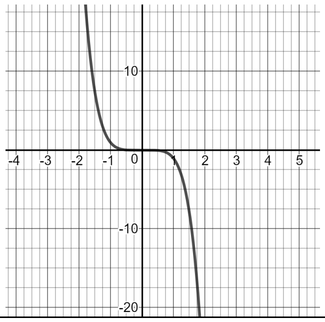
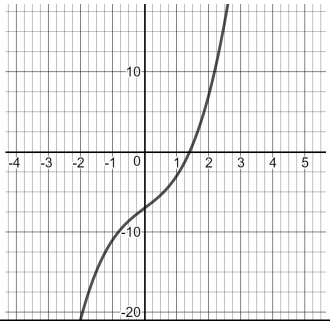
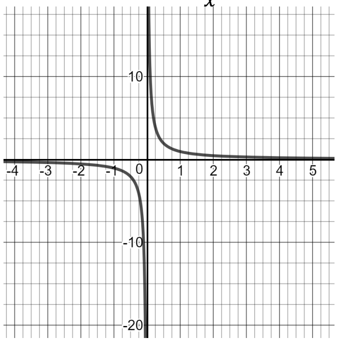
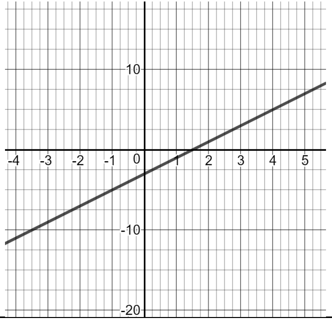
3. The algebraic definition of an even function is:

   $f(x)$ is an even function if and only if $f(x) = f(-x)$ for all values of x in the domain of f.

   What are the implications of the definition for the graph of an even function?

4. Are all even-degree polynomials even functions? Use examples to explain your answer.

5. Let's try the same approach to figure out a definition for odd functions.

<table>
<tr><td>Odd function:
$f(x) = x^3$
 </td><td>Not an odd function:
$g(x) = \log_2 x$
 </td></tr>
<tr><td colspan="2">Differences:</td></tr>
<tr><td>Odd function:
$f(x) = -x^5$
 </td><td>Not an odd function:
$g(x) = x^3 + 3x - 7$
 </td></tr>
<tr><td colspan="2">Differences:</td></tr>
<tr><td>Odd function: $f(x) = \frac{1}{x}$
 </td><td>Not an odd function:
$g(x) = 2x - 3$
 </td></tr>
<tr><td colspan="2">Differences:</td></tr>
<tr><td>Odd function:
$f(2) = 3 \; and \; f(-2) = -3$</td><td>Not an odd function:
$g(2) = 3 \; and \; g(-2) = 5$</td></tr>
<tr><td colspan="2">Differences:</td></tr>
</table>

6. What do you observe about the characteristics of an odd function?

7. The algebraic definition of an odd function is:

$f(x)$ is an odd function if and only if $f(-x) = -f(x)$ for all values of x in the domain of f.

Explain how each of the examples of odd functions above meet this definition.

8. How can you tell if an odd-degree polynomial is an odd function?

9. Are all functions either odd or even?

READY, SET, GO! Name Period Date

READY

Topic: Recognizing special products

Multiply.

1. $(x+5)(x+5)$

2. $(x-3)(x-3)$

3. $(a+b)(a+b)$

4. In problems 1 – 3 the answers are called **perfect square trinomials**. What about these answers makes them be a **perfect square trinomial**?

5. $(x+8)(x-8)$

6. $(x+\sqrt{3})(x-\sqrt{3})$

7. $(x+b)(x-b)$

8. The products in problems 5 – 7 end up being binomials, and they are called the **difference of two squares**. What about these answers makes them be the **difference of two squares**?

Why don't they have a middle term like the problems in 1 – 3?

9. $(x-3)(x^2+3x+9)$

10. $(x+10)(x^2-10x+100)$

11. $(a+b)(a^2-ab+b^2)$

12. The work in problems 9 – 11 makes them feel like the answers are going to have a lot of terms. What happens in the work of the problem that makes the answers be binomials?

These answers are called the **difference of two cubes** (#9) and the **sum of two cubes** (#10 and #11.) What about these answers makes them be the **sum or difference of two cubes**?

SET

Topic: Determining values of polynomials at zero and at ±∞. (End behavior)

State the y-intercept, the degree, and the end behavior for each of the given polynomials.

13. $f(x) = x^5 + 7x^4 - 9x^3 + x^2 - 13x + 8$

y- intercept:

Degree:

End behavior:

As $x \to -\infty$, $f(x) \to$ ________

As $x \to +\infty$, $f(x) \to$ ________

14. $g(x) = 3x^4 + x^3 + 5x^2 - x - 15$

y- intercept:

Degree:

End behavior:

As $x \to -\infty$, $g(x) \to$ ________

As $x \to +\infty$, $g(x) \to$ ________

15. $h(x) = -7x^9 + x^2$

y- intercept:

Degree:

End behavior:

As $x \to -\infty$, $h(x) \to$ ________

As $x \to +\infty$, $h(x) \to$ ________

16. $p(x) = 5x^2 - 18x + 4$

y- intercept:

Degree:

End behavior:

As $x \to -\infty$, $p(x) \to$ ________

As $x \to +\infty$, $p(x) \to$ ________

17. $q(x) = x^3 - 94x^2 - x - 20$

y- intercept:

Degree:

End behavior:

As $x \to -\infty$, $q(x) \to$ ________

As $x \to +\infty$, $q(x) \to$ ________

18. $y = -4x + 12$

y- intercept:

Degree:

End behavior:

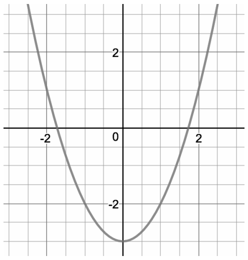
As $x \to -\infty$, $y \to$ ________

As $x \to +\infty$, $y \to$ ________

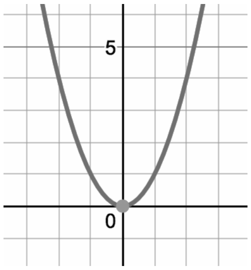
Topic: Identifying even and odd functions

19. Identify each function as even, odd, or neither.

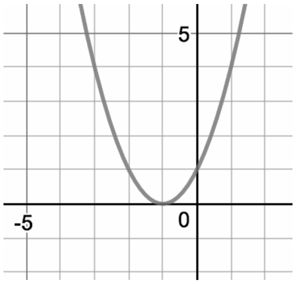
a) $f(x) = x^2 - 3$



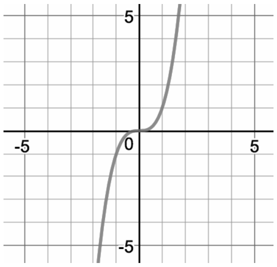
b) $f(x) = x^2$



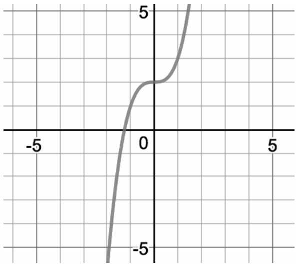
c) $f(x) = (x + 1)^2$



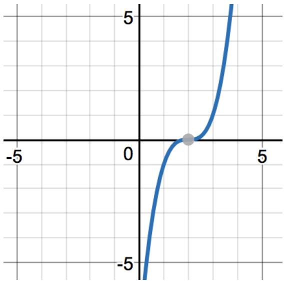
d) $f(x) = x^3$



e) $f(x) = x^3 + 2$



f) $f(x) = (x - 2)^3$



GO

Topic: Factoring special products

Fill in the blanks on the sentences below.

20. The expression $\boldsymbol{a^2 + 2ab + b^2}$ is called a **perfect square trinomial**. I can recognize it because the first and last terms will always be perfect ________________________________.

The middle term will be 2 times the ______________________ and _______________________.

There will always be a _____________ sign before the last term.

It factors as (______________)(______________).

21. The expression $\boldsymbol{a^2 - b^2}$ is called the **difference of 2 squares**. I can recognize it because it's a binomial and the first and last terms are perfect ______________________________.
The sign between the first term and the last term is always a __________________ .
It factors as (___________)(___________).

22. The expression $\boldsymbol{a^3 + b^3}$ is called the **sum of 2 cubes**. I can recognize it because it's a binomial and the first and last terms are ______________________. The expression $a^3 + b^3$ factors into a binomial and a trinomial. I can remember it as a *short* (____) and a *long* (__________).
The sign between the terms in the binomial is the ____________ as the sign in the expression. The first sign in the trinomial is the _______________ of the sign in the binomial. That's why all of the middle terms cancel when multiplying.
The last sign in the trinomial is always _______.
It factors as (___________)(_____________________).

Factor using what you know about special products.

23. $25x^2 + 30 + 9$

24. $x^2 - 16$

25. $x^3 + 27$

26. $49x^2 - 36$

27. $x^3 - 1$

28. $64x^2 - 240 + 225$

3.10 Puzzling Over Polynomials

A Practice Understanding Task

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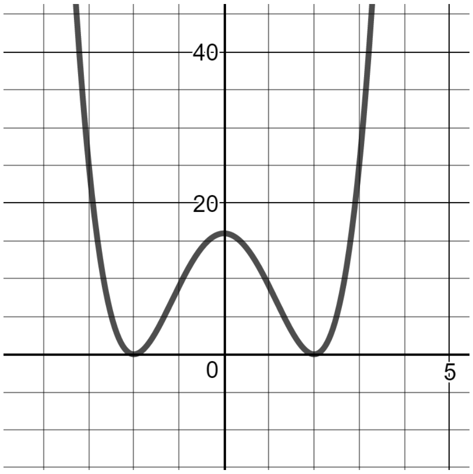
For each of the polynomial puzzles below, a few pieces of information have been given. Your job is to use those pieces of information to complete the puzzle. Occasionally, you may find a missing piece that you can fill in yourself. For instance, although some of the roots are given, you may decide that there are others that you can fill in.

1.	**Function (in factored form)** **Function (in standard form)** **End behavior:** $as\ \ x \to -\infty, \quad f(x) \ \to$ ____ $as\ \ x \to \ \infty, \quad f(x) \ \to$ ____ **Roots (with multiplicity):** -2, 1, and 1 **Value of leading co-efficient:** -2 **Degree:** 3	Graph:

2.	**Function (in factored form)** **Function (in standard form)** **End behavior:** $as \quad x \to -\infty, \qquad f(x) \ \to$ ____ $as \quad x \to \quad \infty, \qquad f(x) \ \to$ ____ **Roots (with multiplicity):** $2 + i, 4, 0$ **Value of leading co-efficient:** 1 **Degree:** 4	Graph:

3.	**Function:** $f(x) = 2(x-1)(x+3)^2$ **End behavior:** $as \quad x \to -\infty, \qquad f(x) \ \to$ ____ $as \quad x \to \quad \infty, \qquad f(x) \ \to$ ____ **Roots (with multiplicity):** **Value of leading co-efficient:** **Domain:** **Range:** All Real numbers	Graph:

4.	**Function:** **End behavior:** $as\ \ x \to -\infty, \quad f(x)\ \to\ \infty$ $as\ \ x \to\ \ \infty, \quad f(x)\ \to$ ____ **Roots (with multiplicity):** (3,0) m: 1; (-1,0) m: 2 (0,0) m: 2 **Value of leading co-efficient:** -1 **Domain:** **Range:**	Graph:
5.	**Function:** **End behavior:** $as\ \ x \to -\infty, \quad f(x)\ \to$ ____ $as\ \ x \to\ \ \infty, \quad f(x)\ \to$ ____ **Roots (with multiplicity):** **Value of leading co-efficient:** 1 **Domain:** **Range:** **Other:** $f(0) = 16$	Graph: 

6.

Function (in standard form):
$f(x) = x^3 - 2x^2 - 7x + 2$

Function (in factored form):

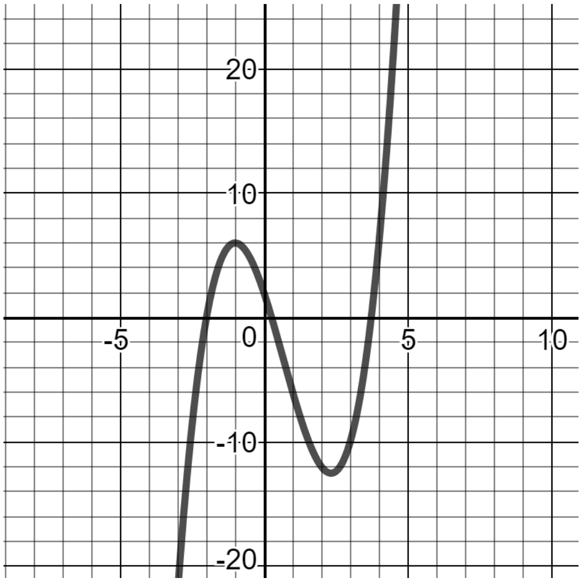
End behavior:
$as \quad x \to -\infty, \qquad f(x) \to$ ____
$as \quad x \to \ \ \infty, \qquad f(x) \to$ ____

Roots (with multiplicity):
-2

Domain:

Range:

Graph:



7.

Function (in standard form):
$f(x) = x^3 - 2x$

Function (in factored form):

End behavior:
$as \quad x \to -\infty, \qquad f(x) \to$ ____
$as \quad x \to \ \ \infty, \qquad f(x) \to$ ____

Roots (with multiplicity):

Domain:

Range:

Graph:

READY, SET, GO! Name Period Date

READY

Topic: Reducing rational numbers and expressions

Reduce the expressions to lowest terms. (Assume no denominator equals 0.)

1. $\frac{3x}{6x^2}$

2. $\frac{2\cdot 5\cdot x\cdot x\cdot x\cdot y}{3\cdot 5\cdot x\cdot y\cdot y}$

3. $\frac{7ab^2}{7ab^2}$

4. $\frac{(x+2)(x-9)}{(x+2)(x-9)}$

5. $\frac{(3x-5)(x+4)}{(x-1)(3x-5)}$

6. $\frac{(2x-11)(3x+17)}{(2x-11)(3x-5)}$

7. $\frac{(8x-7)(x+3)}{8x(x+3)(2x-3)}$

8. $\frac{3x(2x+7)(x-1)(6x-5)}{x(2x+7)(x-1)(6x-5)}$

9. Why is it important that the instructions say to assume that no denominator equals 0?

SET

Topic: Reviewing features of polynomials

Some information has been given for each polynomial. Fill in the missing information.

10.

Function: $f(x) = x^3$

Graph:

Function in factored form:

End behavior:

As $x \to -\infty, f(x) \to$ ______ As $x \to \infty, f(x) \to$ _____

Roots (with multiplicity):

Degree:

Value of leading co-efficient:

11. **Graph:**

Function in standard form:

Function in factored form: $g(x) = -x(x-2)(x-4)$

End behavior:
As $x \to -\infty, g(x) \to$ ______ As $x \to \infty, g(x) \to$ _____

Roots (with multiplicity):

Degree:

Value of leading co-efficient:

12. **Graph:**

Function in standard form: $h(x) = x^3 - 2x^2 - 3x$

Function in factored form:

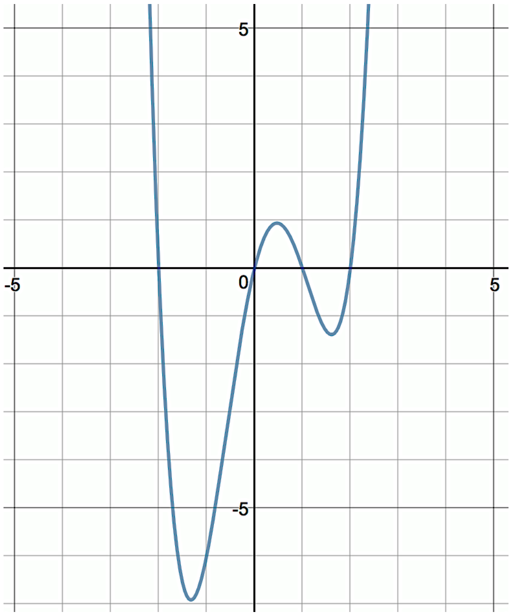
End behavior:
As $x \to -\infty, h(x) \to$ ______ As $x \to \infty, h(x) \to$ _____

Roots (with multiplicity):

Degree:

Value of $h(2)$:

13. **Graph:**



Function in standard form:

Function in factored form:

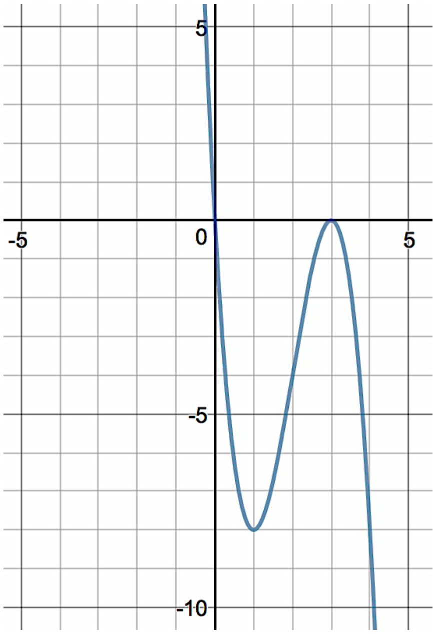
End behavior:
As $x \to -\infty, f(x) \to$ ______ As $x \to \infty, f(x) \to$ _____

Roots (with multiplicity):

Degree:

y-intercept:

14. **Graph:**



Function in standard form:

Function in factored form:

End behavior:
As $x \to -\infty, p(x) \to$ ______ As $x \to \infty, p(x) \to$ _____

Roots (with multiplicity):

Degree:

Value of leading coefficient:

15. **Graph:**

Function in standard form: $q(x) = x^3 + 2x^2 + x + 2$

Function in factored form:

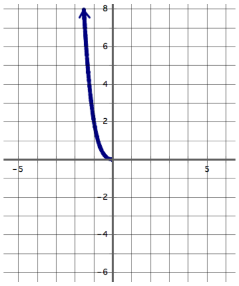
End behavior:
As $x \to -\infty, q(x) \to$ ______ As $x \to \infty, q(x) \to$ _____

Roots (with multiplicity):
$x = i$

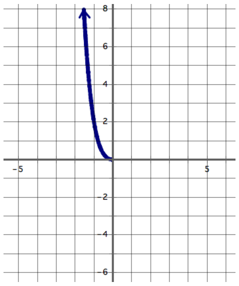
Degree:

y-intercept:

16. Finish the graph if it is an even function.



17. Finish the graph if it is an odd function.



GO

Topic: Writing polynomials given the zeros and the leading coefficient

Write the polynomial function in standard form given the leading coefficient and the zeros of the function.

18. Leading coefficient: 2; $roots$: $2, \sqrt{2}, -\sqrt{2}$

19. Leading coefficient: -1; $roots$: $1, 1+\sqrt{3},\ 1-\sqrt{3}$

20. Leading coefficient: 2; $roots$: $4i, -4i$

Fill in the blanks to make a true statement.

21. If $f(b) = 0$, then a factor of $f(b)$ must be ______________________________.

22. The rate of change in a linear function is always a ______________________________.

23. The rate of change of a quadratic function is ______________________________________.

24. The rate of change of a cubic function is ____.

25. The rate of change of a polynomial function of degree n can be described by a function of degree

____________________.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