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Transforming Mathematics Education

SECONDARY MATH THREE

An Integrated Approach

Standard Teacher Notes

MODULE 1

Functions & Their Inverses

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The Mathematics Vision Project

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FUNCTIONS AND THEIR INVERSES

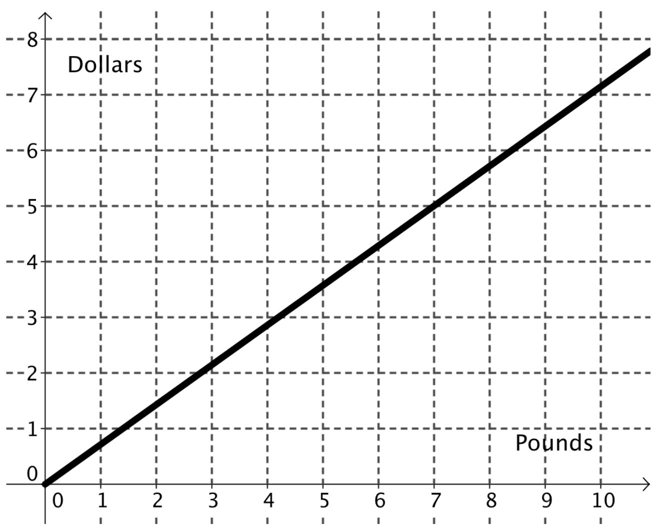
1.1 Brutus Bites Back

A Develop Understanding Task

Remember Carlos and Clarita? A couple of years ago, they started earning money by taking care of pets while their owners are away. Due to their amazing mathematical analysis and their loving care of the cats and dogs that they take in, Carlos and Clarita have made their business very successful. To keep the hungry dogs fed, they must regularly buy Brutus Bites, the favorite food of all the dogs.

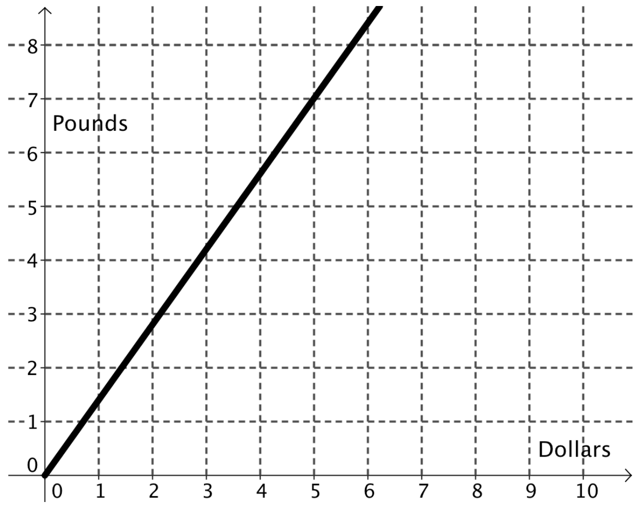
Carlos and Clarita have been searching for a new dog food supplier and have identified two possibilities. The Canine Catering Company, located in their town, sells 7 pounds of food for $5.

Carlos thought about how much they would pay for a given amount of food and drew this graph:



1. Write the equation of the function that Carlos graphed.

Clarita thought about how much food they could buy for a given amount of money and drew this graph:



2. Write the equation of the function that Clarita graphed.

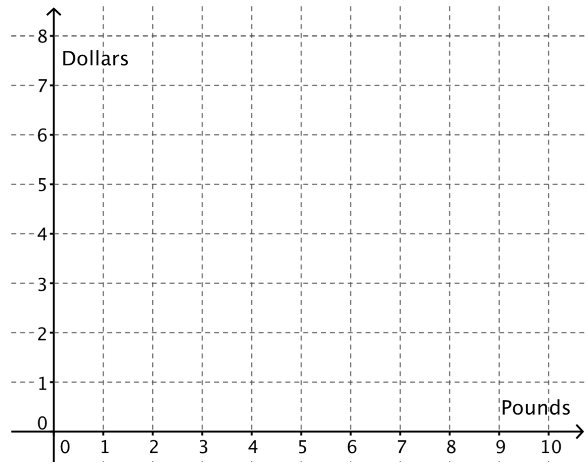
3. Write a question that would be most easily answered by Carlos' graph. Write a question that would be most easily answered by Clarita's graph. What is the difference between the two questions?

4. What is the relationship between the two functions? How do you know?

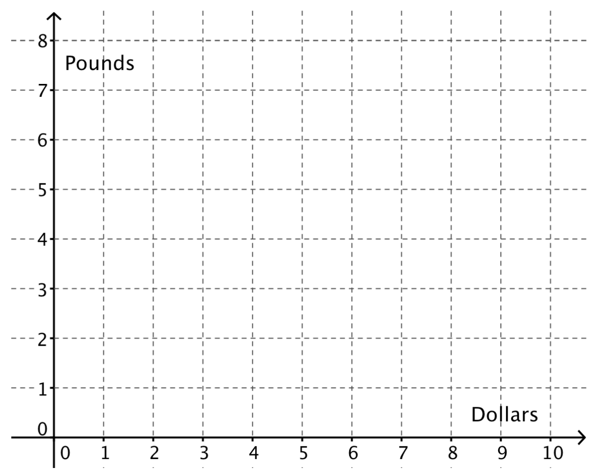
5. Use function notation to write the relationship between the functions.

Looking online, Carlos found a company that will sell 8 pounds of Brutus Bites for $6 plus a flat $5 shipping charge for each order. The company advertises that they will sell any amount of food at the same price per pound.

6. Model the relationship between the price and the amount of food using Carlos' approach.



7. Model the relationship between the price and the amount of food using Clarita's approach.



8. What is the relationship between these two functions? How do you know?

9. Use function notation to write the relationship between the functions.

10. Which company should Clarita and Carlos buy their Brutus Bites from? Why?

1.1 Brutus Bites Back – Teacher Notes

A Develop Understanding Task

Purpose: The purpose of this task is to develop and extend the concept of inverse functions in a linear context. In the task, students use tables, graphs, and equations to represent inverse functions as two different ways of modeling the same situation. The representations expose the idea that the domain of the function is the range of the inverse (and vice versa) for suitably restricted domains. Students may also notice that the graphs of the inverse functions are reflections over the $y = x$ line, but with the understanding that the axes are part of the reflection.

Core Standards Focus:

F.BF.1 Write a function that describes a relationship between two quantities**.**

F.BF.4 Find inverse functions.

a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse.

Related Standards:

F-BF.4.b, F.BF.4c

Standards for Mathematical Practice:

SMP 1 – Make sense of problems and persevere in solving them

SMP 7 – Look for and make use of structure

Vocabulary: Inverse function

The Teaching Cycle:

Launch (Whole Class): Begin the task by ensuring that students understand the context. If they have not done the "Pet Sitters" unit from Secondary Mathematics I, they may not be familiar with the story of Carlos and Clarita, two students that started a business caring for pets. The important part of the context is simply that they are planning to buy dog food that is priced by the pound. Ask students to begin by working individually on questions 1, 2, and 3. Circulate around the room looking at how students are working with the graphs. Identify students to present to the class that have constructed a table of values to aid in writing the equation. Also watch to see the variables that students are using as they write equations. If they are using x as the independent variable for both equations, then they have not properly defined the variable. If this is a common error, you may choose to discuss it with the class.

Once students have written their equations, ask a student to present a table for #1. Identify the difference between terms in the table and then have a student present an equation that uses variables like p (pounds) and d (dollars) and writes an equation for dollars as a function of pounds.

Then have a student present a table of values for #2. Ask the class to compare the table for #1 to the table for #2. They should notice that the p and d column have been switched, making d the independent variable and p the dependent variable. They should also notice the change in the difference between terms on the two tables. Ask for a student to present the equation written for #2. Then, discuss question #3. Be sure that students understand that Carlos thinks about how much it will cost for a certain number of pounds, while Clarita thinks about how much food she can buy with a certain amount of money.

Ask the class to respond to question #4. They should have some experience with inverse functions from Secondary Mathematics II, so they should recognize them in this context. Model the notation for #5 as: $D(p) = P^{-1}\ and\ P(d) = \ D^{-1}$. Ask students for the domain and range of each function.

Explore (Small Group): At this point, let students work together on the remaining problems. As you monitor students as they work, encourage the use of tables to support their writing of equations.

Discuss (Whole Class): Proceed with the discussion much like the previous discussion during the launch. Present the table, graph, and equation for #6 and then the same process for #7. Be sure that students can relate their equations back to the context. For instance, be sure that they can identify why the 5 is added or subtracted in the equations. Help students to see how the structure of the equation of the function is reversed in the inverse function one step at a time. Tie this work to the story context to give it meaning. Again, ask students to compare the domain and range for the functions. Be sure that they notice that the domain of the function is the range of the inverse and vice versa.

Note: Problem #10 is an extension problem for students that may be ahead of the class in completing the task. It is intended to give students an opportunity for additional analysis, but is not critical for the development of the ideas of inverse in this module.

Aligned Ready, Set, Go: *Functions and Their Inverses 1.1*

READY, SET, GO! Name Period Date

READY

Topic: Inverse operations

Inverse operations *"undo"* each other. For instance, addition and subtraction are inverse operations. So are multiplication and division. In mathematics, it is often convenient to *undo* several operations in order to solve for a variable.

Solve for x in the following problems. Then complete the statement by identifying the operation you used to "undo" the equation.

1. 24 = 3x   Undo multiplication by 3 by ______________________________
2. $\frac{x}{5} = -2$ Undo division by 5 by ______________________________
3. $x + 17 = 20$ Undo add 17 by ______________________________
4. $\sqrt{x} = 6$ Undo the square root by ______________________________
5. $\sqrt[3]{(x+1)} = 2$ Undo the cube root by ________________then______________
6. $x^4 = 81$ Undo raising x to the 4th power by ______________________________
7. $(x-9)^2 = 49$ Undo squaring by ________________then______________

SET

Topic: Linear functions and their inverses

Carlos and Clarita have a pet sitting business. When they were trying to decide how many each of dogs and cats they could fit into their yard, they made a table based on the following information. Cat pens require 6 ft^2 of space, while dog runs require 24 ft^2. Carlos and Clarita have up to 360 ft^2 available in the storage shed for pens and runs, while still leaving enough room to move around the cages. They made a table of all of the combinations of cats and dogs they could use to fill the space. They quickly realized that they could fit in 4 cats in the same space as one dog.

cats	0	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60
dogs	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

8. Use the information in the table to write 5 ordered pairs that have cats as the input value and dogs as the output value.

9. Write an explicit equation that shows how many dogs they can accommodate based on how many cats they have. (The number of dogs "d" will be a function of the number of cats "c" or $d = f(c)$.)

10. Use the information in the table to write 5 ordered pairs that have dogs as the input value and cats as the output value.

11. Write an explicit equation that shows how many cats they can accommodate based on how many dogs they have. (The number of cats "c" will be a function of the number of dogs "d" or $c = g(d)$.)

Base your answers in #12 and #13 on the table at the top of the page.

12. Look back at problem 8 and problem 10. Describe how the ordered pairs are different.

13. a) Look back at the equation you wrote in problem 9. Describe the domain for $d = f(c)$.

    b) Describe the domain for the equation $c = g(d)$ that you wrote in problem 11.

    c) What is the relationship between them?

GO

Topic: Using function notation to evaluate a function.

The functions $f(x), g(x), and\ h(x)$ are defined below.

$f(x)=x$ $\qquad g(x)=5x-12$ $\qquad h(x)=x^2+4x-7$

Calculate the indicated function values in the following problems. Simplify your answers.

14. $f(10)$ 15. $f(-2)$ 16. $f(a)$ 17. $f(a+b)$

18. $g(10)$ 19. $g(-2)$ 20. $g(a)$ 20. $g(a+b)$

22. $h(10)$ 23. $h(-2)$ 24. $h(a)$ 25. $h(a+b)$

1.2 Flipping Ferraris

A Solidify Understanding Task

When people first learn to drive, they are often told that the faster they are driving, the longer it will take to stop. So, when you're driving on the freeway, you should leave more space between your car and the car in front of you than when you are driving slowly through a neighborhood. Have you ever wondered about the relationship between how fast you are driving and how far you travel before you stop, after hitting the brakes?

1. Think about it for a minute. What factors do you think might make a difference in how far a car travels after hitting the brakes?

There has actually been quite a bit of experimental work done (mostly by police departments and insurance companies) to be able to mathematically model the relationship between the speed of a car and the braking distance (how far the car goes until it stops after the driver hits the brakes).

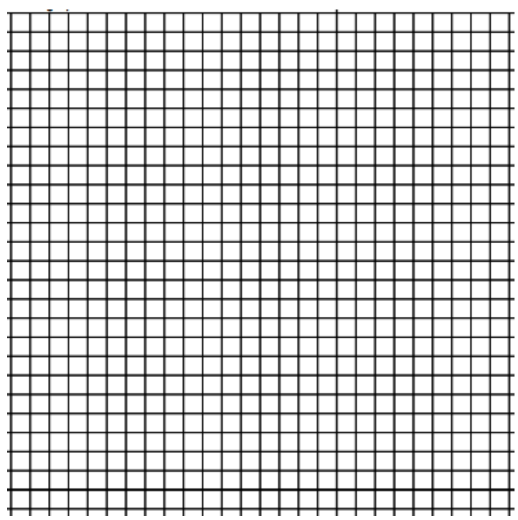
2. Imagine your dream car. Maybe it is a Ferrari 550 Maranello, a super-fast Italian car. Experiments have shown that on smooth, dry roads, the relationship between the braking distance (d) and speed (s) is given by $d(s) = 0.03s^2$. Speed is given in miles/hour and the distance is in feet.
   a) How many feet should you leave between you and the car in front of you if you are driving the Ferrari at 55 mi/hr?

   b) What distance should you keep between you and the car in front of you if you are driving at 100 mi/hr?

   c) If an average car is about 16 feet long, about how many car lengths should you have between you and that car in front of you if you are driving 100 mi/hr?

d) It makes sense to a lot of people that if the car is moving at some speed and then goes twice as fast, the braking distance will be twice as far. Is that true? Explain why or why not.

3. Graph the relationship between braking distance *d(s),* and speed *(s)*, below.



4. According to the Ferrari Company, the maximum speed of the car is about 217 mph. Use this to describe all the mathematical features of the relationship between braking distance and speed for the Ferrari modeled by $d(s) = 0.03s^2$.

5. What if the driver of the Ferrari 550 was cruising along and suddenly hit the brakes to stop because she saw a cat in the road? She skidded to a stop, and fortunately, missed the cat. When she got out of the car she measured the skid marks left by the car so that she knew that her braking distance was 31 ft.

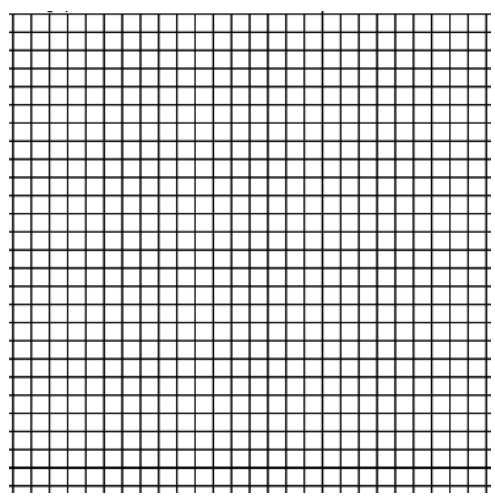
   a) How fast was she going when she hit the brakes?

   b) If she didn't see the cat until she was 15 feet away, what is the fastest speed she could be traveling before she hit the brakes if she wants to avoid hitting the cat?

6. Part of the job of police officers is to investigate traffic accidents to determine what caused the accident and which driver was at fault. They measure the braking distance using skid marks and calculate speeds using the mathematical relationships just like we have here, although they often use different formulas to account for various factors such as road conditions. Let's go back to the Ferrari on a smooth, dry road since we know the relationship. Create a table that shows the speed the car was traveling based upon the braking distance.

7. Write an equation of the function *s(d)* that gives the speed the car was traveling for a given braking distance.

8. Graph the function *s(d)* and describe its features.



9. What do you notice about the graph of *s(d)* compared to the graph of *d(s)*? What is the relationship between the functions *d(s)* and *s(d)*?

10. Consider the function $d(s) = 0.03s^2$ over the domain of all real numbers, not just the domain of this problem situation. How does the graph change from the graph of *d(s)* in question #3?

11. How does changing the domain of *d(s)* change the graph of the inverse of *d(s)*?

12. Is the inverse of *d(s)* a function? Justify your answer.

1.2 Flipping Ferraris – Teacher Notes

A Solidify Understanding Task

Special Note to Teachers: Students may be interested to know that the information about the Ferrari provided in this task is based upon actual studies performed to test the cars at various speeds. The units for speed and distance were changed from metric to the more familiar imperial units so that students could use their intuitive knowledge to think about the reasonableness of their answers.

Purpose: The purpose of this task is to extend students' understanding of inverse functions to include quadratic functions and square roots. In the task, students are given the equation of a quadratic function for braking distance and asked to think about the distance needed to stop safely for a given speed. The logic is then turned around and students are asked to model the speed the car was going for a given braking distance. After students have worked with the inverse relationships with the limited domain [0, 217] (assuming that 217 is the maximum speed of the car), then students are asked to consider the quadratic function and its inverse on the entire domain. This is to elicit a discussion of when and under what conditions the inverse of a function is also a function.

Core Standards Focus:

F.BF.1 Write a function that describes a relationship between two quantities**.**

F.BF.4 Find inverse functions.

c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.
d. (+) Produce an invertible function from a non-invertible function by restricting the domain.

Standards for Mathematical Practice:

SMP 2 – Reason abstractly and quantitatively

SMP 7 – Look for and make use of structure

Vocabulary: Invertible function

The Teaching Cycle:

Launch (Whole Class):

Begin the discussion by sharing student ideas about question #1. Some factors that they may include are: the kind of tires on the car, the type of road surface, the angle of the road surface, the weather, the speed of the car, etc. It's not important to identify which factors actually are considered; the question is simply to foster thinking about the problem situation. Explain to students that if you are driving you want to be sure that there is plenty of room between you and the car in front of you to be sure that you don't hit the car if you both have to stop suddenly. Ask students to work problems 1-4.

Explore (Small Group):

Monitor students as they are working to see that they are not spending too much time on the initial questions, especially question d. The purpose of question d is to think about the rate of change in a quadratic relationship versus a linear relationship. Many people maintain linear thinking patterns even when they know the relationship is not linear. Also, watch to see that students are identifying and labeling an appropriate scale for the graph of the function in this context.

Discuss (Whole Class):

When students have completed their work on #4, begin the discussion with the graph of $d(s)$. Ask students to describe the mathematical features of the graph including: domain, range, minimum value, x and y-intercepts, increasing or decreasing, etc. Ask students to use the graph to provide evidence for their answers to question #3. Then, direct students to complete the task.

Explore (Small Group):

Monitor students as they work to be sure they are setting up tables and applying appropriate scales to their graphs. Listen for statements that can be shared about the inverse relationships that are highlighted in this process. They should be noticing that the independent and dependent variables are being switched and how that is appearing in the table and the graph. They have not been asked to write an equation to model the graph, although many students may identify that it is a square root function. When students get to questions 10 and 11, they may need a little help in interpreting the question.

Discuss (Whole Class):

Begin much like the earlier discussion with the table, graph and features of $s(d)$. Highlight comments from the students that have recognized the inverse relationship between the two functions and articulate the switch between the dependent and independent variables. Show that this makes the inverse function a reflection over the $y = x$ line (including the axes). Be sure that you ask students to compare the domain and range of each of the functions as part of the justification of the inverse relationship.

Shift the discussion to question #10 and ask students to add the additional part of the graph of $d(s)$ that would be included if the domain is extended to all real numbers. Ask students how that would change the graph of $s(d)$. Would $s(d)$ be a function? Students should be familiar with parabolas that have a horizontal line of symmetry, and to be able to articulate why they are not functions. Ask students how they might be able to use the graph of a function to tell if its inverse will be a function. This should bring up some intuitive thinking about one-to-one functions and the idea that if a function has the same y-value for more than one x-value then the inverse will not be a function. Show students how the domain was limited in the case of $d(s)$ to make the function invertible (so that the inverse is a function). You may wish to show a few function graphs and ask students how they might select a domain so that the function is invertible.

Aligned Ready, Set, Go: ***Functions and Their Inverses 1.2***

READY, SET, GO! Name Period Date

READY

Topic: Solving for a variable

Solve for *x*.

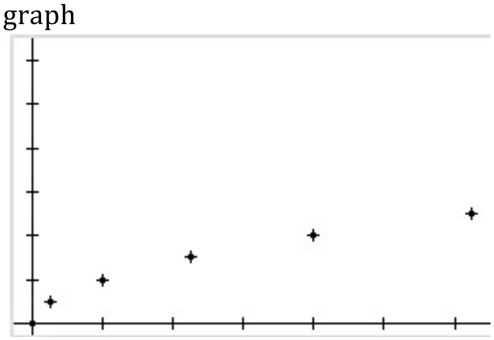
1. $17 = 5x + 2$
2. $2x^2 - 5 = 3x^2 - 12x + 31$
3. $11 = \sqrt{2x+1}$
4. $\sqrt{x^2 + x - 2} = 2$
5. $-4 = \sqrt[3]{5x+1}$
6. $\sqrt[3]{352} = \sqrt[3]{7x^2 + 9}$
7. $9^x = 243$
8. $5^x = \frac{1}{125}$
9. $4^x = \frac{1}{32}$

SET

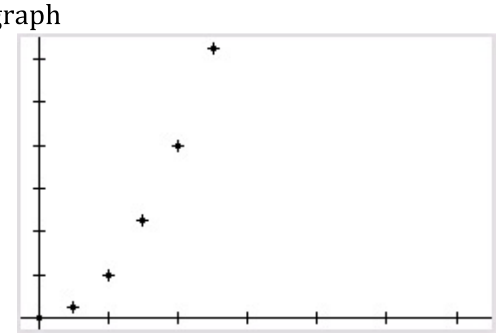
Topic: Exploring inverse functions

10. Students were given a set of data to graph. After they had completed their graphs, each student shared his graph with his shoulder partner. When Ethan and Emma saw each other's graphs, they exclaimed together, "Your graph is wrong!" Neither graph is wrong. Explain what Ethan and Emma have done with their data.

Ethan's graph



Emma's graph



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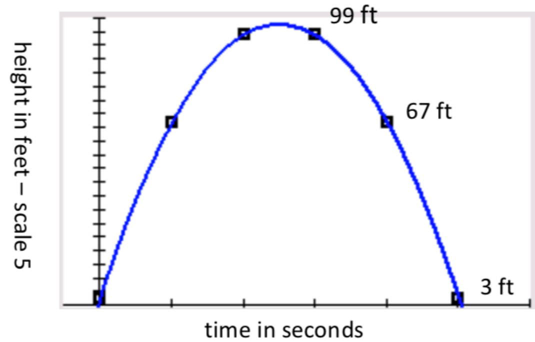
11. Describe a sequence of transformations that would take Ethan's graph onto Emma's.

12. A baseball is hit upward from a height of 3 feet with an initial velocity of 80 feet per second (about 55 mph). The graph shows the height of the ball at any given second during its flight. Use the graph to answer the questions below.



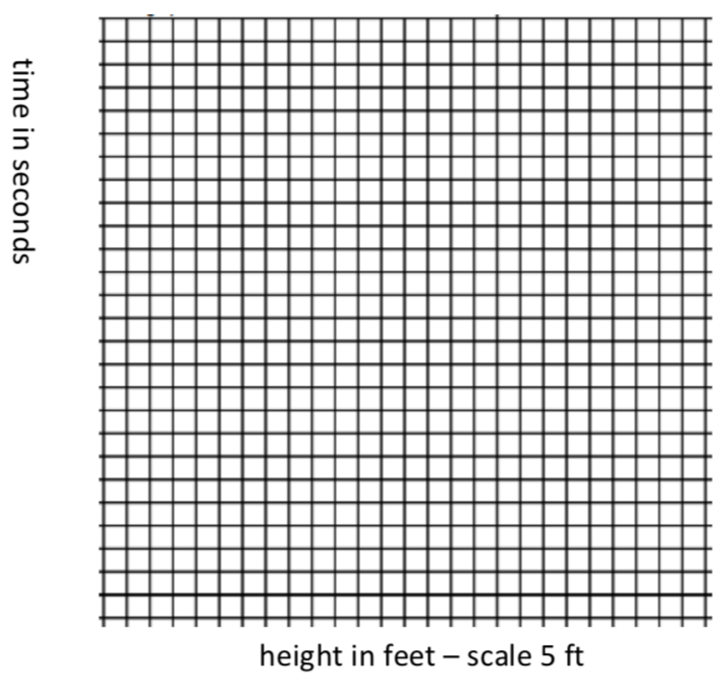
a. Approximate the time that the ball is at its maximum height.

b. Approximate the time that the ball hits the ground.

c. At what time is the ball 67 feet above the ground?

d. Make a new graph that shows the time when the ball is at the given heights.

e. Is your new graph a function? Explain.



GO

Topic: Using function notation to evaluate a function

The functions $f(x), g(x), and\ h(x)$ are defined below.

$f(x) = 3x$ $g(x) = 10x + 4$ $h(x) = x^2 - x$

Calculate the indicated function values. Simplify your answers.

13. $f(7)$
14. $f(-9)$
15. $f(s)$
16. $f(s-t)$
17. $g(7)$
18. $g(-9)$
19. $g(s)$
20. $g(s-t)$
21. $h(7)$
22. $h(-9)$
23. $h(s)$
24. $h(s-t)$

Notice that the notation *f(g(x))* is indicating that you replace *x* in *f(x)* with *g(x).*

Simplify the following.

25. *f(g(x))*
26. *f(h(x))*
27. *g(f(x))*

1.3 Tracking the Tortoise

A Solidify Understanding Task

You may remember a task from last year about the famous race between the tortoise and the hare. In the children's story of the tortoise and the hare, the hare mocks the tortoise for being slow. The tortoise replies, "Slow and steady wins the race." The hare says, "We'll just see about that," and challenges the tortoise to a race.

In the task, we modeled the distance from the starting line that both the tortoise and the hare travelled during the race. Today we will consider only the journey of the tortoise in the race.

Because the hare is so confident that he can beat the tortoise, he gives the tortoise a 1 meter head start. The distance from the starting line of the tortoise including the head start is given by the function:

$d(t) = 2^t$ (d in meters and t in seconds)

The tortoise family decides to watch the race from the sidelines so that they can see their darling tortoise sister, Shellie, prove the value of persistence.

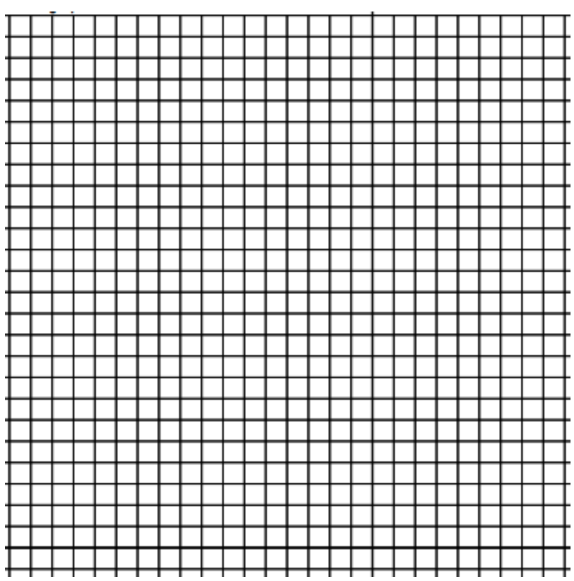
1. How far away from the starting line must the family be, to be located in the right place for Shellie to run by 5 seconds after the beginning of the race? After 10 seconds?

2. Describe the graph of $d(t)$, Shellie's distance at time t. What are the important features of $d(t)$?

3. If the tortoise family plans to watch the race at 64 meters away from Shellie's starting point, how long will they have to wait to see Shellie run past?

4. How long must they wait to see Shellie run by if they stand 1024 meters away from her starting point?

5. Draw a graph that shows how long the tortoise family will wait to see Shellie run by at a given location from her starting point.

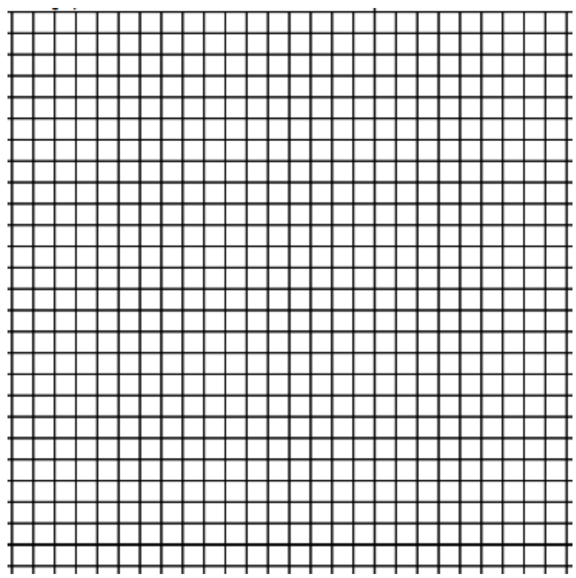


6. How long must the family wait to see Shellie run by if they stand 220 meters away from her starting point?

7. What is the relationship between $d(t)$ and the graph that you have just drawn? How did you use $d(t)$ to draw the graph in #5?

8. Consider the function $f(x) = 2^x$.
   A) What are the domain and range of $f(x)$? Is $f(x)$ invertible?

   B) Graph $f(x)$ and $f^{-1}(x)$ on the grid below.



   C) What are the domain and range of $f^{-1}(x)$?

9. If $f(3) = 8$, what is $f^{-1}(8)$? How do you know?

10. If $f\left(\frac{1}{2}\right) = 1.414$, what is $f^{-1}(1.414)$? How do you know?

11. If $f(a) = b$ what is $f^{-1}(b)$? Will your answer change if *f(x)* is a different function? Explain.

1. 3 Tracking The Tortoise – Teacher Notes

A Solidify Understanding Task

Purpose:

The purpose of this task is two-fold:

1. To extend the ideas of inverse to exponential functions.
2. To develop informal ideas about logarithms based upon understanding inverses. (These ideas will be extended and formalized in Module II: Logarithmic Functions)

The first part of the task builds on earlier work in Secondary I and II with exponential functions. Students are given an exponential context, the distance travelled for a given time, and then asked to reverse their thinking to consider the time travelled for a given distance. After creating a graph of the situation, students are asked to consider the extended domain of all real numbers and think about how that affects the inverse function. Questions at the end of the task press on understanding of the inverse relationship and the idea that a function and its inverse "undo" each other, using function notation and particular values of a function and its inverse.

Core Standards Focus:

F.BF.1 Write a function that describes a relationship between two quantities**.**

F.BF.4 Find inverse functions.

c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.

F.BF.5 (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

Standards for Mathematical Practice:

SMP 3 – Construct viable arguments and critique the reasoning of others

SMP 8 – Look for and express regularity in repeated reasoning

Vocabulary: logarithm (The word should only be introduced in this task as the name of the function that is the inverse of an exponential function. The definition of "logarithm" will be formalized and extended in Module 2.)

The Teaching Cycle:

Launch (Whole Class):

Begin the task by being sure that students understand the problem situation. (If students have done Module I in Secondary Mathematics II, they will be familiar with the context.) Ask students to work individually on problems 1 and 2. Briefly discuss the answers to question 1 and be sure that students know how to use the model given. Then, ask students to describe what they know about *d(t)* before they graph the function. Be sure to include features such as: continuous, y-intercept (0,1), and always increasing. Use technology to draw the graph and confirm their predictions about the graph.

Explore (Small Group):

Have students work on the rest of the task. Questions 3 and 4 help to draw them into the inverse context, much like the work in the previous two tasks. If students are initially stuck, ask them what strategies they used to draw the graph in Flipping Ferraris. Encourage the use of tables to generate points on the graph. Students should use their graph to estimate a value for question #6—they do not yet have experience with logarithms to solve an equation of this type.

Questions 8-11 become more abstract and rely on their previous work with inverses. You may wish to start the discussion when most students have finished question #7 and discuss problems 3-7, before directing students to work on 8-11 and then discussing those problems.

Discuss (Whole Class):

Begin the discussion by asking a student to present a table of values for the graph of *t(d)*. Ask students to describe how they selected the values of *d* to elicit the idea that they chose powers of 2 because they were easy to think about the time. This strategy will become increasingly important in Module II, so it is important to make it explicit here. Draw the graph of the inverse function *t(d)* and ask students how it compares to the graph of *d(t)*. They should again notice the reflection over $y = x$. Ask students to compare the domain and range of *d(t)* with the domain and range of *t(d).* As students discuss the graphs and conclude that *t(d)* is the inverse function of *d(t),* ask them to begin to generalize the patterns they see about inverses to all functions. Some questions you could ask are:

- Will a function and its inverse always be reflections over the $y = x$ line? Why?
- Will the domain of the function always be the range of the inverse function? Why or why not?

Move the discussion to question #8. It is important in this early exposure to the ideas of logarithms that students consider the part of a logarithmic graph between $x = 0$ and $x = 1$. As you ask students to draw $f^{-1}(x)$, consider some particular values in this region. Use the strategy of finding powers of 2 to generate values for $x = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$. Use their answers from the previous part of the discussion to extend the graph and justify the vertical asymptote at $x = 0$. Tell students that this is a new kind of function that will be explored in depth later, but it is called a logarithmic function. This function is written: $y = \log_2 x$. (No need to spend too much time on the notation. It will be solidified later.)

Complete the discussion with questions 9-10. Ask students to show how they can use the graphs of $f(x)$ and $f^{-1}(x)$ to justify their answers. Emphasize question 11 and ask students to make a general argument based on the three tasks and their knowledge of inverses so far. End the discussion by introducing a more formal definition of inverse functions as follows:

In mathematics, an inverse function is a function that "reverses" or "undoes" another function. To describe this relationship in symbols, we say, "The function g is the inverse of function f if and only if $f(a) = b$ and $g(b) = a$. Using x and y, we would write $f(x) = y$ and $g(y) = x$.

Aligned Ready, Set, Go: ***Functions and Their Inverses 1.3***

READY, SET, GO! Name Period Date

READY

Topic: Solving exponential equations.

Solve for the value of *x*.

1. $5^{x+1} = 5^{2x-3}$
2. $7^{3x-2} = 7^{-2x+8}$
3. $4^{3x} = 2^{2x-8}$
4. $3^{5x-4} = 9^{2x-3}$
5. $8^{x+1} = 2^{2x+3}$
6. $3^{x+1} = \frac{1}{81}$

SET

Topic: Exploring the inverse of an exponential function

In the fairy tale *Jack and the Beanstalk*, Jack plants a magic bean before he goes to bed. In the morning Jack discovers a giant beanstalk that has grown so large, it disappears into the clouds.

But here is the part of the story you never heard. Written on the bag containing the magic beans was this note.

Plant a magic bean in rich soil just as the sun is setting. Do not look at the plant site for 10 hours. (This is part of the magic.) After the bean has been in the ground for 1 hour, the growth of the sprout can be modeled by the function $b(t) = 3^t$. (b in feet and t in hours)

Jack was a good math student, so although he never looked at his beanstalk during the night, he used the function to calculate how tall it should be as it grew. The table on the right shows the calculations he made every half hour.

Hence, Jack was not surprised when, in the morning, he saw that the top of the beanstalk had disappeared into the clouds.

Time (hours)	Height (feet)
1	3
1.5	5.2
2	9
2.5	15.6
3	27
3.5	46.8
4	81
4.5	140.3
5	243
5.5	420.9
6	729
6.5	1,262.7
7	2,187
7.5	3,788
8	6,561
8.5	11,364
9	19,683
9.5	34,092
10	59,049

7. Demonstrate how Jack used the model $b(t) = 3^t$ to calculate how high the beanstalk would be after 6 hours had passed. (You may use the table but write down where you would put the numbers in the function if you didn't have the table.)

8. During that same night, a neighbor was playing with his drone. It was programmed to hover at 243 ft. How many hours had the beanstalk been growing when it was as high as the drone?

9. Did you use the table in the same way to answer #8 as you did to answer #7? Explain.

10. While Jack was making his table, he was wondering how tall the beanstalk would be after the magical 10 hours had passed. He quickly typed the function into his calculator to find out. Write the equation Jack would have typed into his calculator.

11. Commercial jets fly between 30,000 ft. and 36,000 ft. About how many hours of growing could pass before the beanstalk might interfere with commercial aircraft? Explain how you got your answer.

12. Use the table to find $f(7)$ and $f^{-1}(11{,}364)$.

13. Use the table to find $f(9)$ and $f^{-1}(9)$.

13. Explain why it's possible to answer some of the questions about the height of the beanstalk by just plugging the numbers into the function rule and why sometimes you can only use the table.

GO

Topic: Evaluating functions

The functions $f(x), g(x), and\ h(x)$ are defined below.

$f(x) = -2x$ $g(x) = 2x + 5$ $h(x) = x^2 + 3x - 10$

Calculate the indicated function values. Simplify your answers.

14. $f(a)$
15. $f(b^2)$
16. $f(a + b)$
17. $f(g(x))$
18. $g(a)$
19. $g(b^2)$
20. $g(a + b)$
21. $h(f(x))$
22. $h(a)$
23. $h(b^2)$
24. $h(a + b)$
25. $h(g(x))$

1.4 Pulling a Rabbit Out of the Hat

A Solidify Understanding Task

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I have a magic trick for you:

- Pick a number, any number.
- Add 6
- Multiply the result by 2
- Subtract 12
- Divide by 2
- The answer is the number you started with!

People are often mystified by such tricks but those of us who have studied inverse operations and inverse functions can easily figure out how they work and even create our own number tricks. Let's get started by figuring out how inverse functions work together.

For each of the following function machines, decide what function can be used to make the output the same as the input number. Describe the operation in words and then write it symbolically.

Here's an example:

Input		Output
$x = 7$ →	7 + 8 = 15 →	7

$f(x) = x + 8$

$\boxed{f^{-1}(x) = x - 8}$

In words: Subtract 8 from the result

1.

Input				Output
$x = 7$	$\rightarrow$	$3 \cdot 7 = 21$	$\rightarrow$	7

$f(x) = 3x$

$\boxed{f^{-1}(x) = \quad}$

In words:

2.

Input				Output
$x = 7$	$\rightarrow$	$7^2 = 49$	$\rightarrow$	7

$f(x) = x^2$

$\boxed{f^{-1}(x) = \quad}$

In words:

3.

Input				Output
$x = 7$	$\rightarrow$	$2^7 = 128$	$\rightarrow$	7

$f(x) = 2^x$

$\boxed{f^{-1}(x) = \quad}$

In words:

4. Input Output

$x = 7 \rightarrow 2 \cdot 7 - 5 = 9 \rightarrow 7$

$f(x) = 2x - 5$

$\boxed{f^{-1}(x) = \qquad}$

In words:

5. Input Output

$x = 7 \rightarrow \dfrac{7+5}{3} = 4 \rightarrow 7$

$f(x) = \dfrac{x+5}{3}$

$\boxed{f^{-1}(x) = \qquad}$

In words:

6. Input Output

$x = 7 \rightarrow (7-3)^2 = 16 \rightarrow 7$

$f(x) = (x-3)^2$

$\boxed{f^{-1}(x) = \qquad}$

In words:

7. Input Output

$x = 7$ → $4 - \sqrt{7}$ → 7

$f(x) = 4 - \sqrt{x}$

$\boxed{f^{-1}(x) = \quad}$

In words:

8. Input Output

$x = 7$ → $2^7 - 10 = 118$ → 7

$f(x) = 2^x - 10$

$\boxed{f^{-1}(x) = \quad}$

In words:

9. Each of these problems began with $x = 7$. What is the difference between the x used in $f(x)$ and the x used in $f^{-1}(x)$?

10. In #6, could any value of x be used in $f(x)$ and still give the same output from $f^{-1}(x)$? Explain. What about #7?

11. Based on your work in this task and the other tasks in this module what relationships do you see between functions and their inverses?

1. 4 Pulling A Rabbit Out Of The Hat – Teacher Notes

A Solidify Understanding Task

Purpose: The purpose of this task is to solidify students' understanding of the relationship between functions and their inverses and to formalize writing inverse functions. In the task, students are given a function and a particular value for input value x, and then asked to describe and write the function that that will produce an output that is the original x value. The task relies on students' intuitive understanding of inverse operations such as subtraction "undoing" addition or square roots "undoing" squaring. There are two exponential problems where students can describe "undoing" an exponential function and the teacher can support the writing of the inverse function using logarithmic notation.

Core Standards Focus:

F.BF.4. Find inverse functions.

a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. For example, $f(x) = 2x^3$ or $f(x) = (x+1)/(x-1)$ for $x \neq 1$.
b. (+) Verify by composition that one function is the inverse of another.

Standards for Mathematical Practice:

SMP 6 – Attend to precision

SMP 7 – Look for and make use of structure

The Teaching Cycle:

Launch (Whole Class):

Begin class by having students try the number trick at the beginning of the task. After they try it with their own number, help them to track through the operations to show why it works as follows:

- Pick a number x
- Add 6 $x + 6$
- Multiply the result by 2 $2(x + 6) = 2x + 12$
- Subtract 12 $2x + 12 - 12 = 2x$
- Divide by 2 $\frac{2x}{2} = x$
- The answer is the number you started with! x

This should highlight the idea that inverse operations "undo" each other. A function may involve more than one operation, so if the inverse function is to "undo" the function, it may have more than one operation and those operations may need to be performed in a particular order. Tell students that in this task, they will be finding inverse functions, which will be described in words and then symbolically. Work through the example with the class and then let them talk with their partners or group about the rest of the problems.

Explore (Small Group):

Monitor students as they work to see that they are making sense of the inverse operations and considering the order that is needed on the functions that require two steps. Encourage them to describe the operations in the correct order before they write the inverse function symbolically. Because the notation for logarithmic functions has barely been introduced in the previous task, students may not know how to write the inverse function for #2 and #8. Tell them that is acceptable as long as they have described the operation for the inverse in words. Accept informal expressions like, "undo the exponential", but challenge students that may say that the inverse of the exponential is some kind of root, like an "x^{th} root".

As you listen to students talking about the problems, find one or two problems that are generating controversy or misconceptions to discuss with the entire class.

Discuss (Whole Class):

Begin the discussion with problems #4 and #6. Ask students to describe the inverse function in words and then help the class to write the inverse function. Then support students in using log notation for #3 with the following statements:

$2^7 = 128$ $\log_2 128 = 7$

$f(x) = 2^x$ $f^{-1}(x) = \log_2 x$

Discuss some of the problems that generated controversy or confusion during the Explore phase.

End the discussion by challenging students to write the expression for #8. Before working on the notation, ask students to describe the inverse operations and decide how the order has to go to properly unwind the function. They should say that you need to add 10 and then undo the exponential. Give them some time to think about how to use notation to write that and then ask students to offer ideas. They should have seen from previous problems that the +10 needs to go into the argument of the function because it needs to happen before you undo the exponential. So, the notation should be:

$f(x) = 2^x - 10$ $f^{-1}(x) = \log_2(x + 10)$

$2^7 - 10 = 118$ $\log_2(118 + 10) = 7$ (because $2^7 = 128$)

Make sure that there is time left to discuss questions 9, 10 and 11. For question #9 and 10, the main point to highlight is the idea that the output of the function becomes the input for the inverse and vice versa. This is why the domain and range of the two functions are switched (assuming suitable values for each). Press students to make a general argument that this would be true for any function and its inverse.

Question #11 is an opportunity to solidify all the ideas about inverse that have been explored in the unit before the practice task. Some ideas that should emerge:

- A function and its inverse undo each other.
- There are inverse operations like addition/subtraction, multiplication/division, squaring/square rooting. Functions and their inverses use these operations together and they need to be in the right order.
- For a function to be invertible, the inverse must also be a function. (That means that the original function must be one-to-one.)
- The domain of a function can be restricted to make it invertible.

- A function and its inverse look like reflections over the $y = x$ line. (Be careful of this statement because of the way the axes change units for this to be true.)
- The domain (suitably-restricted) of a function is the range of the inverse function and vice versa.

Finalize the discussion of features of inverse functions by introducing a more formal definition of inverse functions as follows:

> In mathematics, an inverse function is a function that "reverses" or "undoes" another function. To describe this relationship in symbols, we say, "The function g is the inverse of function f if and only if $f(a) = b$ and $g(b) = a$. Using x and y, we would write $f(x) = y$ and $g(y) = x$.

If you choose, you can close the class with one more number puzzle for students to figure out on their own:

- Pick a number
- Add 2
- Square the result
- Subtract 4 times the original number
- Subtract 4 from that result
- Take the square root of the number that is left
- The answer is the number you started with.

Aligned Ready, Set, Go: *Functions and Their Inverses 1.4*

READY, SET, GO! Name Period Date

READY

Topic: Properties of exponents

Use the product rule or the quotient rule to simplify. Leave all answers in exponential form with only positive exponents.

1. $3^6 \cdot 3^5$
2. $7^2 \cdot 7^6$
3. $10^{-4} \cdot 10^7$
4. $5^9 \cdot 5^{-6}$
5. $p^2 p^5$
6. $2^6 \cdot 2^{-3} \cdot 2$
7. $b^{11} b^{-5}$
8. $\frac{7^5}{7^2}$
9. $\frac{9^8}{9}$
10. $\frac{3^5}{3^8}$
11. $\frac{7^{-4}}{7^{-8}}$
12. $\frac{p^{-3}}{p^5}$

SET

Topic: Inverse function

13. Given the functions $f(x) = \sqrt{x} - 1$ *and* $g(x) = x^2 + 7$:

a. Calculate $f(16)$ *and* $g(3)$.

b. Write $f(16)$ as an ordered pair.

c. Write $g(3)$ as an ordered pair.

d. What do your ordered pairs for $f(16)$ and $g(3)$ imply?

e. Find $f(25)$.

f. Based on your answer for $f(25)$, predict $g(4)$.

g. Find $g(4)$. Did your answer match your prediction?

h. Are $f(x)$ *and* $g(x)$ inverse functions? Justify your answer.

Match the function in the first column with its inverse in the second column.

$f(x)$	$f^{-1}(x)$
16. $f(x) = 3x + 5$	a. $f^{-1}(x) = log_5 x$
17. $f(x) = x^5$	b. $f^{-1}(x) = \sqrt[3]{x}$
18. $f(x) = \sqrt[5]{x-3}$	c. $f^{-1}(x) = \frac{x-5}{3}$
19. $f(x) = x^3$	d. $f^{-1}(x) = \frac{x}{3} - 5$
20. $f(x) = 5^x$	e. $f^{-1}(x) = log_3 x$
21. $f(x) = 3(x+5)$	f. $f^{-1}(x) = x^5 + 3$
22. $f(x) = 3^x$	g. $f^{-1}(x) = \sqrt[5]{x}$

GO

Topic: Composite functions and inverses

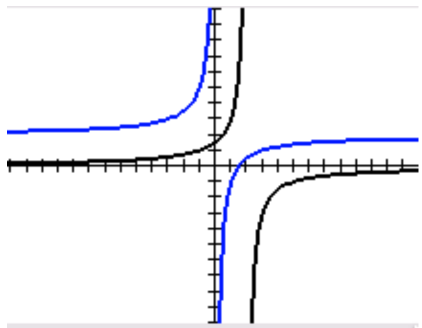
Calculate $f(g(x))$ *and* $g(f(x))$ for each pair of functions.

(Note: the notation $(f \circ g)(x)$ *and* $(g \circ f)(x)$ means the same thing as $f(g(x))$ *and* $g(f(x))$, respectively.)

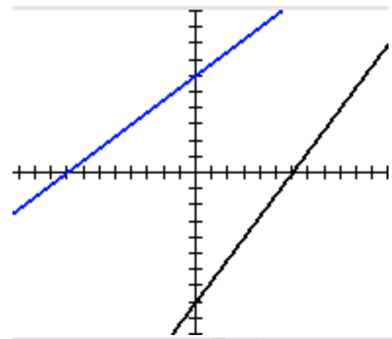
23. $f(x) = 2x + 5$ $\quad g(x) = \frac{x-5}{2}$	24. $f(x) = (x+2)^3$ $\quad g(x) = \sqrt[3]{x} - 2$
25. $f(x) = \frac{3}{4}x + 6$ $\quad g(x) = \frac{4(x-6)}{3}$	26. $f(x) = \frac{-3}{x} + 2$ $\quad g(x) = \frac{-3}{x-2}$

Match the pairs of functions above (23-26) with their graphs. Label *f (x) and g (x)*.

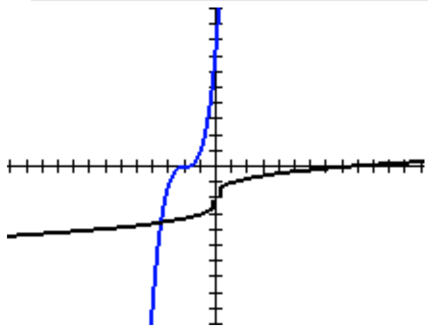
a.



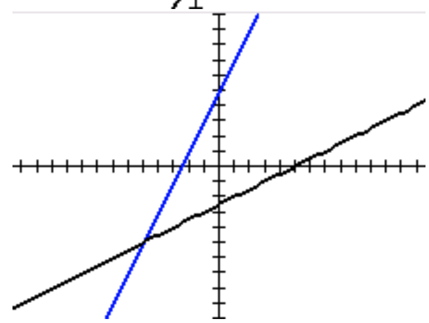
b.



c.



d.



27. Graph the line y = x on each of the graphs above. What do you notice?

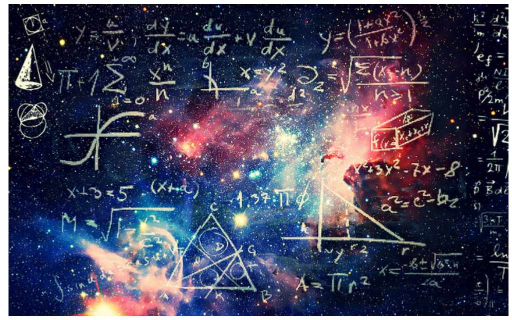
28. Do you think your observations about the graphs in #27 has anything to do with the answers you got when you found $f(g(x))$ *and* $g(f(x))$? Explain.

29. Look at graph *b*. Shade the 2 triangles made by the y-axis, x-axis, and each line. What is interesting about these two triangles?

30. Shade the 2 triangles in graph *d*. Are they interesting in the same way? Explain.

1.5 Inverse Universe

A Practice Understanding Task



You and your partner have each been given a different set of cards. The instructions are:

1. Select a card and show it to your partner.
2. Work together to find a card in your partner's set of cards that represents the inverse of the function represented on your card.
3. Record the cards you selected and the reason that you know that they are inverses in the space below.
4. Repeat the process until all of the cards are paired up.

*For this task only, assume that all tables represent points on a continuous function.

Pair 1: ______________ Justification of inverse relationship: ______________________

Pair 2: ______________ Justification of inverse relationship: ______________________

Pair 3: ______________ Justification of inverse relationship: ______________________

Pair 4: ______________ Justification of inverse relationship: ______________________

Pair 5: ______________ Justification of inverse relationship: ______________________

Pair 6: ________________ Justification of inverse relationship: ____________________________

Pair 6: ________________ Justification of inverse relationship: ____________________________

Pair 7: ________________ Justification of inverse relationship: ____________________________

Pair 8: ________________ Justification of inverse relationship: ____________________________

Pair 9: ________________ Justification of inverse relationship: ____________________________

Pair 10: ________________ Justification of inverse relationship: ____________________________

A1

$$f(x) = \begin{cases} -2x - 2, & -5 < x < 0 \\ -2, & x \geq 0 \end{cases}$$

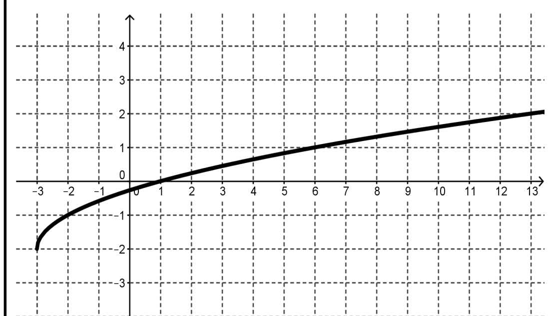
A2

The function increases at a constant rate of $\frac{a}{b}$ and the y-intercept is (0, c).

A3

Each input value, x, is squared and then 3 is added to the result. The domain of the function is [0,∞)

A4



A5

x	y
-2	-3
2	3
0	0
6	5
4	4
$\frac{-4}{3}$	-2

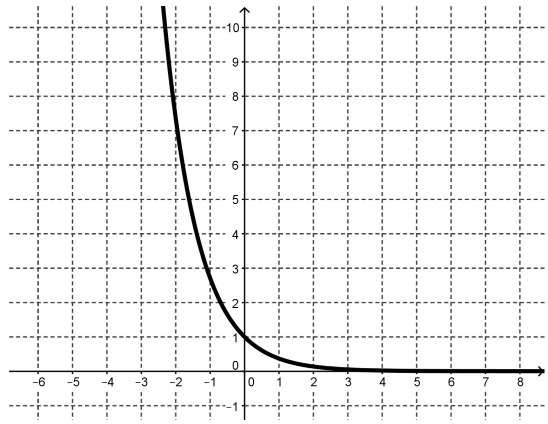
A6

$$y = 3^x$$

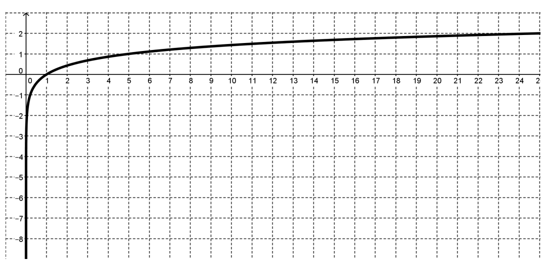
A7

x	y
-5	-125
-3	-27
-1	-1
1	1
3	27
5	125

A8



A9



A10

Yasmin started a savings account with \$5. At the end of each week, she added 3. This function models the amount of money in the account for a given week.

B1

$$y = \log_3 x$$

B2

$$f(x) = \begin{cases} \frac{2}{3}x, & -3 < x < 3 \\ 2x - 4, & x \geq 3 \end{cases}$$

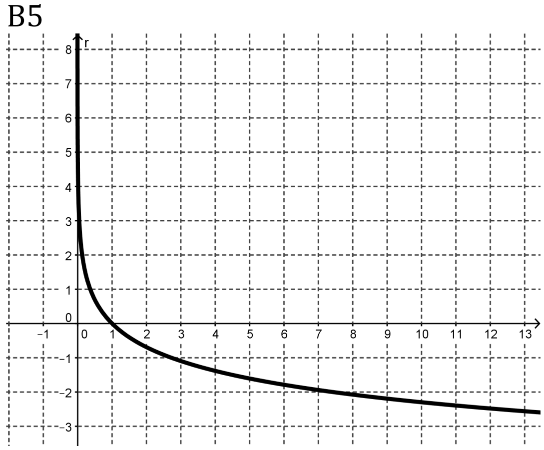
B3

The x-intercept is (c, 0) and the slope of the line is $\frac{b}{a}$.

B4

x	***y***
-216	-6
-64	-4
-8	-2
0	0
8	2
64	4
216	6

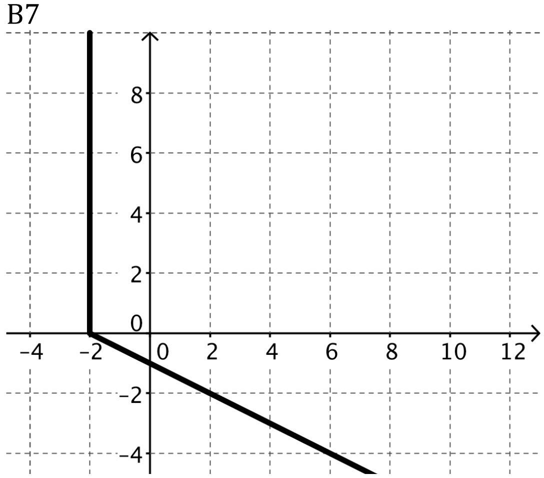
B5



B6

x	***y***
3	0
4	1
7	2
12	3
19	4
28	5
39	6

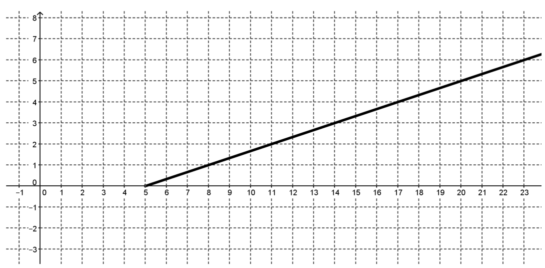
B7



B8

x	y
-2	-3
-1	-2
0	1
1	6
2	13

B9



B10

The function is continuous and grows by an equal factor of 5 over equal intervals. The y-intercept is (0,1).

1.5 Inverse Universe – Teacher Notes

A Practice Understanding Task

Teacher Note: It will be helpful to have the cards used in this task copied and cut before distributing them to students.

Purpose: The purpose of this task is for students to become more fluent in finding inverses and to increase their flexibility in thinking about inverse functions using tables, graphs, equations, and verbal descriptions.

Core Standards Focus:

F.BF.4 Find inverse functions.

a. Solve an equation of the form $f(x) = c$ for a simple function f that has an inverse and write an expression for the inverse. For example, $f(x) = 2x^3$ or $f(x) = (x+1)/(x-1)$ for $x \neq 1$.
b. (+) Verify by composition that one function is the inverse of another.
c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.
d. (+) Produce an invertible function from a non-invertible function by restricting the domain.

Related Standards: F.BF.5

Standards for Mathematical Practice:

SMP 3 – Construct viable arguments and critique the reasoning of others

SMP 6 – Attend to precision

Launch (Whole Class):

Be sure that the entire class understands the instructions. Emphasize that it is important for the discussion that they have fully justified their choices on the recording sheet. Ask the class for ideas about how they can justify that two functions are inverses and record their responses. Tell students to reference the list and to be very specific about the functions that they are working with as they write their justifications. Full justification should include the type of functions that are shown in the representations and how they can show that either the (x, y) values are switched over the entire domain of each of the functions or that the two functions undo each other (by composition—either formally or informally by describing the operations in the necessary order).

Explore (Small Group):

Monitor students as they work, listening for particularly challenging, controversial, or insightful comments generated by the work. Be sure that students are recording their justifications and they are sufficiently specific to the cards they have selected. While listening to students, identify one pair of students to present for each function and its inverse.

Discuss (Whole Class):

If time permits, you may choose to have 10 different pairs of students each demonstrate a pair of functions and how they justified that they were inverses. Look for variety in their justifications, including checking to see if the graphs are reflections, verifying by composition of specific values, or showing how the inputs and outputs have been switched. If there is not enough time to go through each pair, then select the cards that caused the most discussion during the exploration phase. The function pairs are:

A1	B7
A2	B3
A3	B6
A4	B8
A5	B2
A6	B1
A7	B4
A8	B5
A9	B10
A10	B9

Aligned Ready, Set, Go: ***Functions and Their Inverses 1.5***

READY, SET, GO! Name Period Date

READY

Topic: Properties of exponents

Use properties of exponents to simplify the following. Write your answers in exponential form with positive exponents.

1. $\sqrt[7]{x^2} \cdot \sqrt[7]{x^3}$

2. $\sqrt[3]{x} \cdot \sqrt[4]{x} \cdot \sqrt[6]{x}$

3. $\sqrt[6]{a} \cdot \sqrt[3]{a^2} \cdot \sqrt[5]{b^3}$

4. $\sqrt[5]{32} \cdot \sqrt{9} \cdot \sqrt[3]{27}$

5. $\sqrt[4]{8} \cdot \sqrt[3]{16} \cdot \sqrt[6]{2}$

6. $(5^2)^3$

7. $(7^2)^{-1}$

8. $(3^{-4})^{-5}$

9. $\left(\frac{5^{-4}}{5^2}\right)^3$

SET

Topic: Representations of inverse functions

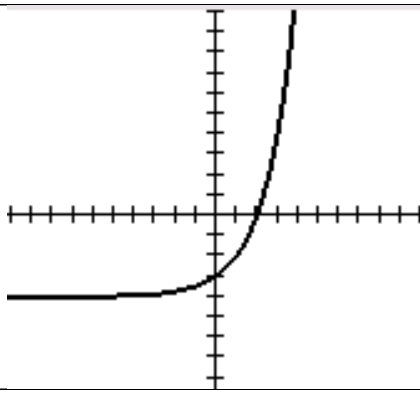
Write the inverse of the given function in the same format as the given function.

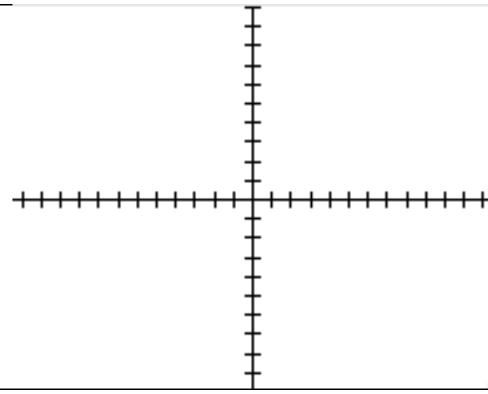
Function $f(x)$	Inverse $f^{-1}(x)$
10.	

x	$f(x)$
-8	0
-4	3
0	6
4	9
8	12

Need help? Visit **www.rsgsupport.org**

11.

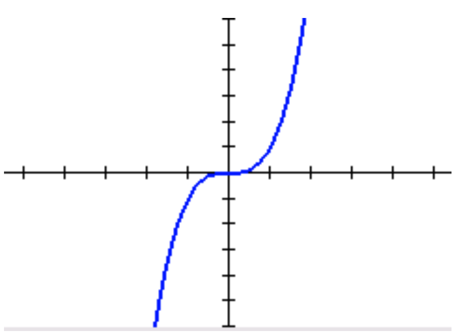


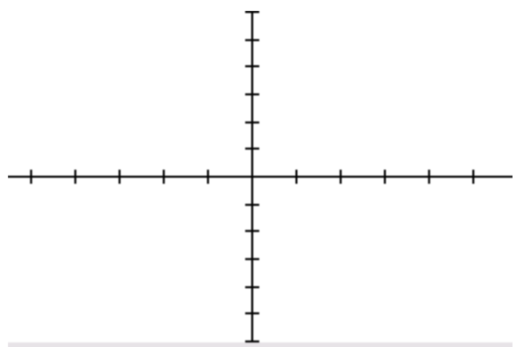


12. $f\ (x) = -2x + 4$

13. $f(x) = log_3x$

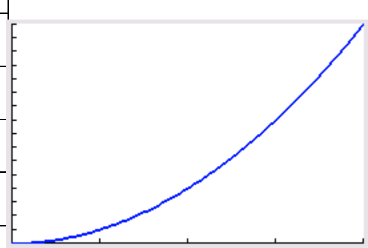
14.

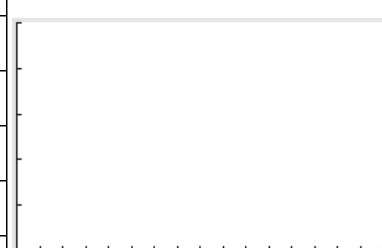




15.

x	$f(x)$
0	0
1	1
2	4
3	9
4	16





GO

Topic: Composite functions

Calculate $f(g(x))$ *and* $g(f(x))$ for each pair of functions.

(Note: the notation $(f \circ g)(x)$ *and* $(g \circ f)(x)$ mean the same thing, respectively.)

16. $f(x) = 3x + 7;\ \ g(x) = -4x - 11$

17. $f(x) = -4x + 60;\ \ g(x) = -\frac{1}{4}x + 15$

18. $f(x) = 10x - 5;\ \ g(x) = \frac{2}{5}x + 3$

19. $f(x) = -\frac{2}{3}x + 4;\ \ g(x) = -\frac{3}{2}x + 6$

20. Look back at your calculations for $f(g(x))$ *and* $g(f(x))$. Two of the pairs of equations are inverses of each other. Which ones do you think they are?
    Why?